

CONCENTRATION-COMPACTNESS PRINCIPLE FOR NONLOCAL SCALAR FIELD EQUATIONS WITH CRITICAL GROWTH

JOÃO MARCOS DO Ó AND DIEGO FERRAZ

ABSTRACT. The aim of this paper is to study a concentration-compactness principle for homogeneous fractional Sobolev space $\mathcal{D}^{s,2}(\mathbb{R}^N)$ for $0 < s < \min\{1, N/2\}$. As an application we establish Palais-Smale compactness for the Lagrangian associated to the fractional scalar field equation $(-\Delta)^s u = f(x, u)$ for $0 < s < 1$. Moreover, using an analytic framework based on $\mathcal{D}^{s,2}(\mathbb{R}^N)$, we obtain the existence of ground state solutions for a wide class of nonlinearities in the critical growth range.

1. INTRODUCTION

The main goal of the present work is to analyze concentration-compactness principles for homogeneous fractional Sobolev spaces. As application, we address questions on compactness of the associated energy functional to the following nonlocal scalar field equation

$$(-\Delta)^s u = f(x, u) \quad \text{in } \mathbb{R}^N, \quad (\mathcal{P}_s)$$

where $0 < s < 1$, and $(-\Delta)^s$ is the fractional Laplacian defined by the relation

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s} \mathcal{F}u(\xi), \quad \xi \in \mathbb{R}^N,$$

where $\mathcal{F}u$ is the Fourier transform of u , i.e.

$$\mathcal{F}u(x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} u(\xi) e^{-i\xi \cdot x} d\xi, \quad x \in \mathbb{R}^N. \quad (1.1)$$

Let \mathcal{S} be the Schwartz space consisting of rapidly decaying C^∞ functions in \mathbb{R}^N which, together with all their derivatives, vanish at the infinity faster than any power of $|x|$. Equivalently, if $u \in \mathcal{S}$ the fractional Laplacian of u can be computed by the following singular integral

$$(-\Delta)^s u(x) = C(N, s) \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(0)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,$$

for a suitable positive normalizing constant

$$C(N, s) = \left(\frac{1 - \cos \varsigma_1}{|\varsigma|^{N+2s}} \right)^{-1}.$$

We refer to [15, 24, 38] for an introduction to the fractional Laplacian operator.

During the past years there has been a considerable amount of research involving nonlocal nonlinear stationary Schrödinger problems. This equation arises in the study of the fractional Schrödinger equation when looking for standing waves. Indeed, when u is a solution of Eq. (\mathcal{P}_s) ,

2000 *Mathematics Subject Classification.* 35J60 (35J20 47J30 49J35 35B33).

Key words and phrases. Scalar field equation, Fractional Laplacian, Concentration-compactness.

Research supported in part by INCTmat/MCT/Brazil, CNPq and CAPES/Brazil.

it can be seen as stationary states (corresponding to solitary waves) in nonlinear equations of Schrödinger type

$$i\phi_t - (-\Delta)^s \phi + f(x, \phi) = 0 \quad \text{in } \mathbb{R}^N.$$

Fractional Schrödinger equations is also of interest in quantum mechanics (see e.g. the appendix in [14] for details and physical motivations). Moreover, we refer to [2], [3] and [10], where equations involving the operator $(-\Delta)^s$ arises from several areas of science such as biology, chemistry or finance.

A lot of work has been devoted to the existence of solutions for nonlinear scalar field equations like Eq. (\mathcal{P}_s) , both for local case ($s = 1$) and nonlocal case $0 < s < 1$, since the celebrated works of H. Berestycki and P.-L. Lions [5, 6]. In these two papers, the authors discuss the existence of radial solutions of the semi-linear elliptic equation

$$-\Delta u = g(u), \quad u \in H^1(\mathbb{R}^N) (N \geq 3), \quad (1.2)$$

where $g : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous odd function with subcritical growth. Under some appropriate conditions on $g(t)$, they used minimizing arguments to prove (in part I) the existence of a positive radial ground state for (1.2), that is, solution having the property of the least action among all possible solutions. In [42], K. Tintarev has treated the non-autonomous problem $-\Delta u = g(x, u)$ in $\mathbb{R}^N (N \geq 3)$, $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, when the nonlinearity $g(x, t)$ is allowed to have critical growth with asymptotically self-similar oscillations about the critical power $|t|^{2^*-2}t$. Recently, using some minimax arguments, X. Chang and Z.-Q. Wang [12] proved the existence of a positive ground state for fractional scalar field equations of the form $(-\Delta)^s u = g(u)$ in $\mathbb{R}^N (N \geq 2)$, $s \in (0, 1)$, when $g(t)$ has subcritical growth and satisfies the Berestycki–Lions type assumptions. In [46], J. M. do Ó et al., established the existence of ground state solutions to the fractional scalar field equation $(-\Delta)^s u = g(u)$ in \mathbb{R}^3 , $s \in (0, 1)$, when $g(t)$ has critical growth.

Motivated by the results cited above, another important purpose of this work is to prove the existence of a ground state solutions for the nonlinear scalar field equation (\mathcal{P}_s) in the “zero mass case” with nonlinearities in the critical growth range. It is well known that Eq. (\mathcal{P}_s) admits a variational setting in fractional Sobolev spaces, and the solutions are constructed with a variational method by a minimax procedure on the associated energy functional. However, we note that the usual variational techniques cannot be applied straightly because of a lack of compactness, which roughly speaking, originates from the invariance of \mathbb{R}^N with respect to translation and dilation and, analytically, appears because of the non-compactness of the Sobolev embedding. For instance, it is not possible to apply the minimax type arguments used by P. Felmer et al. [19] and R. Servadei and E. Valdinoci [36] and [37] because their approach rely strongly on the sub-criticality of the nonlinear terms or the boundedness of the domain. To overcome these difficulties, under appropriate assumptions, we establish a profile decomposition for bounded sequences of suitable Sobolev spaces, proving that Palais-Smale sequences of the associated energy functional converges, up to subsequence.

The idea for proving the existence of ground state solution for Eq. (\mathcal{P}_s) in the autonomous case is based in a constrained minimization argument similar to [5]. We obtain the result by using the invariance of the problem with respect to action of the translation and dilation group in \mathbb{R}^N , thanks to our concentration-compactness principle and a specific Pohozaev identity. Our argument allow us to avoid the typical assumption that $t^{-1}f(x, t)$ is an increasing function, which it is usually required on the approach of constrained minimization over Nehari manifold. Moreover, to prove the existence for the autonomous case $f(x, t) = f(t)$, we do not require the well known Ambrosetti-Rabinowitz condition.

The existence for the general case of Eq. (\mathcal{P}_s) is obtained from the autonomous case which was derived from a class of Pohozaev identity for the space $\mathcal{D}^{s,2}(\mathbb{R}^N)$. The proof of this identity is essentially based in the use of the so called s -harmonic extension introduced by L. Caffarelli and L. Silvestre [11] and remarks contained in [18] and [23].

Our main results may be seen as the nonlocal counterpart of some theorems of K. Tintarev et al. [41–43]. In comparison with the local case [42], we also mention some additional difficulties: the Pohozaev type identities for the fractional framework available in the literature (cf. [12, 20, 35]) do not match with our settings; an additional hypothesis (assumption (f_5)) must be considered in order to achieve the concentration-compactness for the non-autonomous case. In fact, the asymptotic additivity (f_5) takes the role to describe precisely the behavior of weak convergence under our settings (Proposition 7.1). At this point a natural question arises: Is hypothesis (f_5) necessary to describe the limit of the profile decomposition terms (see Theorem 2.1)? Indeed, we believe that without condition (f_5) it is possible to find examples for which this description fails.

To the best of our knowledge, this is the first work that shows a Pohozaev type identity for the homogeneous Sobolev space $\mathcal{D}^{s,2}(\mathbb{R}^N)$ and for $f(t)$ in the critical growth range. Our method is very convenient in the sense that with our arguments we can always derive a Pohozaev type identity in the fractional framework without relying in global regularization of the solutions. In the present literature, there is only Pohozaev type identities for solutions in the inhomogeneous fractional Sobolev space $H^s(\mathbb{R}^N)$, $0 < s < \min\{1, N/2\}$, and for $f(t)$ with subcritical growth (cf. [12]). Moreover, the argument for the proof relies in obtaining the behavior of solutions in the whole space \mathbb{R}^N (cf. [20]).

In addition, we introduce a new class of nonlinearities of the critical growth type for the fractional framework, that include the power $|t|^{2^*_s}$ as an example. We believe that this new notion of criticality together with our concentration-compactness, can lead to a new way to approach elliptic problems involving nonlinearities with critical growth and the fractional Laplacian, for instance, replacing the well known nonlinearity $f(x, t) = K(x)|t|^{2^*-1}t$ for a general *self-similar* function under our settings (see Sect. 3.1).

Moreover, as it is well known, one of the main difficulties in dealing with nonlinearities with critical growth condition is proving that the minimax level of the functional associated to Eq. (\mathcal{P}_s) avoids levels of non-compactness, which usually requires additional description of the nonlinearity growth. We avoid this by considering that $f(x, t)$ has appropriated limits consistent with our concentration-compactness and comparing the minimax level of functional associated to Eq. (\mathcal{P}_s) with the limit ones.

1.1. Outline. The paper is organized as follows. Sect. 2 is devoted to the description of the profile decomposition of bounded sequence in the Sobolev space $\mathcal{D}^{s,2}(\mathbb{R}^N)$. In Sect. 3, we give some applications of Theorem 2.1 to study of existence of mountain-pass solutions of Eq. (\mathcal{P}_s) , for autonomous and non-autonomous case. In Sect. 4, we state some basic results (without prove) on the fractional Sobolev space $\mathcal{D}^{s,2}(\mathbb{R}^N)$. We also prove a Pohozaev identity for weak solutions of Eq. (\mathcal{P}_s) and establish the background material to develop the profile decomposition described in Sect. 2. In Sect. 5, by establishing how dilations and translation acts in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, we prove the abstract result of Sect. 2. In Sect. 6, we study the basic properties for a class of nonlinearities in the critical growth range dealt in this paper, which is fairly used to establish the results regarding the Eq. (\mathcal{P}_s) . In Sect. 7, using the properties obtained in the Sect. 6, we describe the limit of the profile decomposition of the Palais-Smale sequence at the mountain pass level of the Lagrangian of Eq. (\mathcal{P}_s) . In Sect. 8, we prove the results gave in Subsect. 3.2 and describe some properties regarding the minimax levels associated with the functional energy of Eq. (\mathcal{P}_s) , for the

autonomous case. In Sect. 9, we prove our result about the existence of non-trivial weak solution of Eq. (\mathcal{P}_s) in the non-autonomous case, and for the sake of discussion, we establish a sufficient condition that ensures one of our hypothesis, precisely, the assumption (f_8) .

2. PROFILE DECOMPOSITION IN $\mathcal{D}^{s,2}(\mathbb{R}^N)$

For $0 < s < \min\{1, N/2\}$, let us consider the homogeneous fractional Sobolev space $\mathcal{D}^{s,2}(\mathbb{R}^N)$, which is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|^2 := \int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}u|^2 d\xi.$$

It is well known that $\mathcal{D}^{s,2}(\mathbb{R}^N)$ is continuously embedded in $L^{2_s^*}(\mathbb{R}^N)$, $0 < s < \min\{1, N/2\}$, where $2_s^* = 2N/(N - 2s)$. The aforementioned concentration-compactness is made by means of profile decomposition for bounded sequences in homogeneous fractional Sobolev spaces, which can be considered as extensions of the Banach-Alaoglu theorem. This kind of profile decomposition has been widely investigated in various settings, for instance we may cite the ones in [21, 22, 31, 39, 40]. It describes how the convergence of a bounded sequence fails in the considered space. Our approach in this matter is based in the abstract version of profile decomposition in Hilbert spaces due to K. Tintarev and K.-H. Fieseler [43], and it seems for us that this approach is more appropriated to study existence of nontrivial solutions for problems like (\mathcal{P}_s) , under our settings.

Theorem 2.1. *Let $(u_k) \subset \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a bounded sequence, $\gamma > 1$ and $0 < s < \min\{1, N/2\}$. Then there exists $\mathbb{N}_* \subset \mathbb{N}$, disjoint sets (if non-empty) $\mathbb{N}_0, \mathbb{N}_-, \mathbb{N}_+ \subset \mathbb{N}$, with $\mathbb{N}_* = \mathbb{N}_0 \cup \mathbb{N}_+ \cup \mathbb{N}_-$ and sequences $(w^{(n)})_{n \in \mathbb{N}_*} \subset \mathcal{D}^{s,2}(\mathbb{R}^N)$, $(y_k^{(n)})_{k \in \mathbb{N}} \subset \mathbb{Z}^N$, $(j_k^{(n)})_{k \in \mathbb{N}} \subset \mathbb{Z}$, $n \in \mathbb{N}_*$, such that, up to subsequence of (u_k) ,*

$$\gamma^{-\frac{N-2s}{2}j_k^{(n)}} u_k(\gamma^{-j_k^{(n)}} \cdot + y_k^{(n)}) \rightharpoonup w^{(n)} \text{ as } k \rightarrow \infty, \text{ in } \mathcal{D}^{s,2}(\mathbb{R}^N), \quad (2.1)$$

$$|j_k^{(n)} - j_k^{(m)}| + |\gamma^{j_k^{(n)}}(y_k^{(n)} - y_k^{(m)})| \rightarrow \infty, \text{ as } k \rightarrow \infty, \text{ for } m \neq n, \quad (2.2)$$

$$\sum_{n \in \mathbb{N}_*} \|w^{(n)}\|^2 \leq \limsup_{k \rightarrow \infty} \|u_k\|^2, \quad (2.3)$$

$$u_k - \sum_{n \in \mathbb{N}_*} \gamma^{\frac{N-2s}{2}j_k^{(n)}} w^{(n)}(\gamma^{j_k^{(n)}}(\cdot - y_k^{(n)})) \rightarrow 0, \text{ as } k \rightarrow \infty, \text{ in } L^{2_s^*}(\mathbb{R}^N), \quad (2.4)$$

and the series in (2.4) converges uniformly in k . Furthermore, $1 \in \mathbb{N}_0$, $y_k^{(1)} = 0$; $j_k^{(n)} = 0$ whenever $n \in \mathbb{N}_0$; $j_k^{(n)} \rightarrow -\infty$ whenever $n \in \mathbb{N}_-$; and $j_k^{(n)} \rightarrow +\infty$ whenever $n \in \mathbb{N}_+$.

This decomposition is unique up to permutation of indices, and up to constant operator multiples (See [43, Proposition 3.4]). As it can be seen, Theorem 2.1 describes how the convergence of bounded sequences in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ fails to converge in $L^{2_s^*}(\mathbb{R}^N)$. This “error” of convergence is generated, roughly speaking, by the invariance of action of the group of translation and dilation in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Observe that the behavior for the correction term in (2.4) is precisely described in the assertions (2.1)–(2.3).

In [21] and [31] the authors introduced the subject in the fractional framework, and also, the problem of cocompactness in the sense of [13] was extensively discussed. In addition to the study of the above profile decomposition, we develop techniques to obtain non-trivial weak solutions of Eq. (\mathcal{P}_s) , thus, extending the results in [42]. It seems for us that this new abstract result is more appropriated to study the existence of nontrivial solutions for scalar field equation (\mathcal{P}_s) than the profile decomposition developed in [31]. It is not clear how one can apply [31, Theorem 4] to obtain

such a result for nonlinearities with asymptotically self-similar oscillations about the fractional critical growth (see Sect. 6 for precise definitions). It is worth to mention that Theorem 2.1 can be used to prove the fractional version of Lions concentration-compactness principle proved in [31, Theorem 5]. Indeed, Theorem 2.1 improves [31, Theorem 5] for the case $\Omega = \mathbb{R}^N$, since the sums of Dirac masses that appears in this result comes from the profiles given in (2.4).

We point out some differences of our result and the profile decompositions present in the literature [21, 22, 31] as follows. By lowering the range of the fractional space $\mathcal{D}^{s,2}(\mathbb{R}^N)$ to $0 < s < 1$, we were able to improve the translations part from \mathbb{R}^N to \mathbb{Z}^N , allowing to get more information about the dilation as well. From this, considering the specific form of dilations as

$$\delta_j u(x) = \gamma^{\frac{N-2s}{2}j} u(\gamma^j x), \quad j \in \mathbb{Z} \quad \text{instead of} \quad \delta_\lambda u(x) = \lambda^{\frac{N-2s}{2}} u(\lambda x), \quad \lambda > 0,$$

the collection of the “dislocated profiles” $w^{(n)}$, could be decomposed in three: dilation by “enlargement” (\mathbb{N}_-), dilation by “reducement” (\mathbb{N}_+), and no dilation (pure translation \mathbb{N}_0).

3. NONLINEAR SCALAR FIELD EQUATIONS

In this section we state our main results regarding the existence of solutions of Eq. (\mathcal{P}_s). In what follows, we always assume that $0 < s < \min\{1, N/2\}$.

3.1. Hypothesis. In order to describe our results on the energy functional of (\mathcal{P}_s) in a more precisely way, we will make the following assumptions:

$$f : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R} \text{ satisfies the Carathéodory conditions.} \quad (f_1)$$

$$|f(x, t)| \leq C|t|^{2^*_s-1}, \quad \forall t \in \mathbb{R} \text{ and a.e. } x \in \mathbb{R}^N. \quad (f_2)$$

$$\exists \mu > 2; \quad \mu F(x, t) := \mu \int_0^t f(x, \tau) d\tau \leq f(x, t)t, \quad \forall t \in \mathbb{R} \text{ and a.e. } x \in \mathbb{R}^N. \quad (f_3)$$

$$\begin{aligned} &\exists (x_0, t_0) \in \mathbb{R}^N \times \mathbb{R}_+ \text{ such that} \\ &|B_R| \inf_{B_R(x_0)} F(x, t_0) + |B_{R+1} \setminus B_R| \inf_{(x,t) \in (B_{R+1}(x_0) \setminus B_R(x_0)) \times [0, t_0]} F(x, t) > 0 \end{aligned} \quad (f_4)$$

In the study of the autonomous case we consider a weak version of (f_4) which we state next.

$$\exists t_0 > 0 \text{ such that } F(t_0) > 0. \quad (f'_4)$$

For any real numbers $a_1, \dots, a_M, \exists C = C(M) > 0$ such that

$$\left| F\left(x, \sum_{n=1}^M a_n\right) - \sum_{n=1}^M F(x, a_n) \right| \leq C(M) \sum_{m \neq n \in \{1, \dots, M\}} |a_n|^{2^*_s-1} |a_m| \quad \text{a.e. } x \in \mathbb{R}^N. \quad (f_5)$$

$\exists \gamma > 1, 0 < s < \min\{1, N/2\}$ such that the following limits exists

$$\begin{aligned} f_0(t) &:= \lim_{|x| \rightarrow \infty} f(x, t), \\ f_+(t) &:= \lim_{j \in \mathbb{Z}, j \rightarrow +\infty} \gamma^{-\frac{N+2s}{2}j} f\left(\gamma^{-j}x, \gamma^{\frac{N-2s}{2}j}t\right), \\ f_-(t) &:= \lim_{j \in \mathbb{Z}, j \rightarrow -\infty} \gamma^{-\frac{N+2s}{2}j} f\left(\gamma^{-j}x, \gamma^{\frac{N-2s}{2}j}t\right). \end{aligned} \quad (f_6)$$

uniformly in x and in compact sets for t .

f_0, f_+, f_- are continuously differentiable. (f7)

We consider associated with Eq. (\mathcal{P}_s) , the functional $I : \mathcal{D}^{s,2}(\mathbb{R}^N) \rightarrow \mathbb{R}$ given by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 dx - \int_{\mathbb{R}^N} F(x, u) dx. \quad (3.1)$$

Assuming that $f(x, t)$ satisfies (f_2) and using the same arguments of [32], $I \in C^1(\mathcal{D}^{s,2}(\mathbb{R}^N))$ and

$$I'(u) \cdot v = \int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} v dx - \int_{\mathbb{R}^N} f(x, u) v dx, \quad u, v \in \mathcal{D}^{s,2}(\mathbb{R}^N).$$

Thus critical points of I correspond to weak solutions of Eq. (\mathcal{P}_s) and conversely.

Regarding the minimax level, we consider

$$\Gamma_I = \left\{ \gamma \in C([0, \infty), \mathcal{D}^{s,2}(\mathbb{R}^N)) : \gamma(0) = 0, \lim_{t \rightarrow \infty} I(\gamma(t)) = -\infty \right\},$$

and

$$c(I) = \inf_{\gamma \in \Gamma_I} \sup_{t \geq 0} I(\gamma(t)). \quad (3.2)$$

For the nonlinearities f_0, f_+, f_- , we consider the associated energy functionals given by

$$I_\kappa(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 dx - \int_{\mathbb{R}^N} F_\kappa(u) dx, \quad F_\kappa(t) := \int_0^t f_\kappa(\tau) d\tau$$

and the respectively minimax levels

$$c_\kappa = \inf_{\gamma \in \Gamma_\kappa} \sup_{t \geq 0} I_\kappa(\gamma(t))$$

where

$$\Gamma_\kappa = \left\{ \gamma \in C([0, \infty), \mathcal{D}^{s,2}(\mathbb{R}^N)) : \gamma(0) = 0, \lim_{t \rightarrow \infty} I_\kappa(\gamma(t)) = -\infty \right\},$$

for $\kappa = 0, +, -$. Next, we assume a condition that compares the mountain pass levels defined above, precisely, for each $\kappa = 0, +, -$,

$$F_\kappa \text{ satisfies } (f'_4) \quad \text{and} \quad c(I) < c(I_\kappa). \quad (f_8)$$

We also consider the additional assumption for $\kappa = 0, +, -$,

$$F_\kappa \text{ satisfies } (f'_4), \quad F_\kappa(t) \leq F(x, t), \quad \text{a.e. } x \in \mathbb{R}^N, \quad \forall t \in \mathbb{R}. \quad (3.3)$$

$$\text{Moreover } \exists \delta > 0 \text{ such that } F_\kappa(t) < F(x, t), \quad \text{a.e. } x \in \mathbb{R}^N, \quad \forall t \in (-\delta, \delta) \quad (f'_8)$$

We are going to prove in Proposition 9.1 that (f'_8) implies (f_8) . To obtain our main result, we first study the autonomous case. For that we assume:

$$\exists \gamma > 1, \quad 0 < s < \min\{1, N/2\}, \quad \text{such that } G(t) = \gamma^{-Nj} G\left(\gamma^{\frac{N-2s}{2}j} t\right), \quad \forall j \in \mathbb{Z}, \quad \forall t \in \mathbb{R}. \quad (f_9)$$

This allow us to derive some basic results concerning the behavior upon the functional I as we pass the limit over corrected sequences given in Theorem 2.1. We refer to the class of functions that satisfies (f_9) as *self-similar* (see Section 6).

In this paper, we often use the notation $\Phi(u) = \int_{\mathbb{R}^N} F(x, u) dx$ and $\Phi_\kappa(u) = \int_{\mathbb{R}^N} F_\kappa(u) dx$ for $\kappa = 0, +, -$, also, we usually refer to the assumptions (f_1) – (f_8) for the autonomous case $f(x, t) = f(t)$.

3.2. Statement of main results. Next, we state the main result abouts the autonomous case $f(x, t) = f(t)$.

Theorem 3.1. *Suppose that $f(t)$ satisfies (f_1) , (f'_4) and (f_9) , and consider*

$$\mathcal{S}_l = \sup_{\|u\|^2=l} \int_{\mathbb{R}^N} F(u) dx. \quad (3.4)$$

Then, for any maximizing sequence (u_k) of (3.4) there exists $(j_k) \subset \mathbb{Z}$ and $(y_k) \subset \mathbb{Z}^N$ such that $(\gamma^{-\frac{N-2s}{2}j_k} u_k(\gamma^{-j_k} \cdot + y_k))$ contains a convergent subsequence in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. In particular, the supremum in (3.4) is attained. Moreover, the same conclusion holds for

$$\mathcal{S}_{l,+} = \sup_{\|u\|^2=l} \int_{\mathbb{R}^N} F_+(u) dx \quad \text{and} \quad \mathcal{S}_{l,-} = \sup_{\|u\|^2=l} \int_{\mathbb{R}^N} F_-(u) dx,$$

provided that $f(t)$ satisfies (f_1) , (f_2) and (f_6) , with F_+, F_- fulfilling (f'_4) and $\mathcal{S}_1 > \max\{\mathcal{S}_{1,+}, \mathcal{S}_{1,-}\}$.

Our next result proves that maximizers of (3.4) are indeed non-trivial solutions of Eq. (\mathcal{P}_s) . Moreover, the mountain pass level (3.2) is attained. The main tool to achieve these facts is a Pohozaev type identity proved in Section 4.2, which holds under the condition $0 < s < 1$ and taking into account the smoothness of the nonlinearity.

Theorem 3.2. *Assume that $f(t) \in C^1(\mathbb{R})$ satisfies (f_1) , (f_2) and (f'_4) .*

- (i) *If v is a nonzero critical point of I , then $c(I) \leq I(v)$;*
- (ii) *If w is a maximizer of \mathcal{S}_{l_0} for $l_0 := (2_s^* \mathcal{S}_1)^{-\frac{N-2s}{2s}}$, then w is a critical point of I . Moreover*

$$0 < \max_{t \geq 0} I(w(\cdot/t)) = I(w) = c(I).$$

From Theorem 3.2, we conclude that to obtain weak solutions for the autonomous case, only assumptions (f_1) , (f'_4) and (f_9) are needed. Moreover, we are able to prove that the minimax level is attained without the Ambrosetti-Rabinowitz condition (f_3) .

Another way to approach Eq. (\mathcal{P}_s) is by the means of constrained minimization. In fact, due Theorem 2.1 we can argue as [43], and thanks to the Pohozaev identity, reasoning as in [5], we can derive existence of a ground state solution (or least energy) for Eq. (\mathcal{P}_s) , that is, a solution u of (\mathcal{P}_s) such that $I(u) \leq I(v)$, for any other solution v .

Theorem 3.3. *Suppose that $f(t) \in C^1(\mathbb{R}^N)$ satisfies (f_1) , (f'_4) and (f_9) . Let*

$$\mathcal{G} = \{u \in \mathcal{D}^{s,2}(\mathbb{R}^N) : \Phi(u) = 1\}$$

and consider

$$\mathcal{I} = \inf_{u \in \mathcal{G}} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u(x)|^2 dx. \quad (3.5)$$

Then, for any minimizing sequence (u_k) of (3.5) there exists $(j_k) \subset \mathbb{Z}$ and $(y_k) \subset \mathbb{Z}^N$ such that $(\gamma^{-\frac{N-2s}{2}j_k} u_k(\gamma^{-j_k} \cdot + y_k))$ contains a convergent subsequence in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. In particular, there exists a minimizer w for (3.5). Furthermore, $u(x) = w(x/\beta)$ is a ground state solution for Eq. (\mathcal{P}_s) for some $\beta > 0$.

In the following result we prove that Palais-Smale condition at the mountain pass level holds for the general non-autonomous case.

Theorem 3.4. *If $f(x, t)$ satisfies (f_1) – (f_7) and (3.3), then Eq. (\mathcal{P}_s) has a nontrivial weak solution u in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ at the mountain pass level, that is, $I(u) = c(I)$. Moreover, if we assume additionally that (f_8) holds true, then any sequence (u_k) in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ such that $I(u_k) \rightarrow c(I)$ and $I'(u_k) \rightarrow 0$ has a convergent subsequence.*

3.2.1. Remark on the hypothesis.

Remark 3.5. Next we give several helpful comments concerning our assumptions.

- (i) On assumption (f_1) , we recall that $f : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Carathéodory conditions, if for each fixed $t \in \mathbb{R}$, $f(\cdot, t)$ is measurable, and for a.e. $x \in \mathbb{R}^N$, $f(x, \cdot)$ is continuous in \mathbb{R} .
- (ii) Condition (f_2) include in particular nonlinearities with critical growth.
- (iii) Assumption (f_3) is a weak version of the well-known Ambrosetti-Rabinowitz condition in the sense that we do not require that $F(x, t)$ is positive. (see [1, 33]).
- (iv) In order to prove that the functional associated with Eq. (\mathcal{P}_s) has the mountain pass geometry we consider (f_4) . Furthermore, since we deal with constrained minimization, an autonomous version of (f_4') is needed (see [5]).
- (v) The asymptotic additivity given in (f_5) ensure the convergence of functional I under the weak profile decomposition for bounded sequences in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ described in Theorem 2.1.
- (vi) The smoothness condition $f(t) \in C^1(\mathbb{R})$ is the natural hypothesis used in the literature to prove that weak solutions of Eq. (\mathcal{P}_s) satisfies a Pohozaev type identity.
- (vii) Once the limits in (f_6) exist, to obtain compactness of Palais-Smale sequences at the minimax levels we need to require the additional conditions over the minimax levels c_0, c_+, c_- given in assumption (f_8) . In fact, we do not believe that it is possible, in general, to achieve the compactness described in Theorem 3.4 without these conditions. We mention that this kind of approach was introduced by P.-L. Lions in [26–29].
- (viii) Observe that the approach to obtain concentration-compactness for the autonomous case $f(x, t) = f(t)$ need to be different since in this case, $f(t)$ does not satisfies (f_8) .
- (ix) We also consider the case when (f_8) do not hold. Precisely, when it is allowed $c(I) = c(I_\kappa)$, for some $\kappa = 0, +, -$. In this case, the concentration-compactness argument at the mountain pass level cannot be used. We apply [25, Theorem 2.3] to overcome this difficulty and prove existence of solution at the mountain pass level.
- (x) The limits F_\pm fulfills (f_9) . Thus, functions $f(x, t)$ that satisfies (f_6) could be seen as being asymptotically self-similar at $\pm\infty$ (see Remark 6.5).
- (xi) In our results, one can assume that $f(x, t) = f(|x|, t)$ is a radial function in x , instead of the existence of the asymptote $f_0(t)$. This fact can be easily verified considering Proposition ??.
- (xii) In Lemma 8.3 we proved that $\Gamma_\kappa \neq \emptyset$ is equivalent to: $\exists t_\kappa$ such that $F_\kappa(t_\kappa) > 0$.

Remark 3.6. We have that $\mathcal{G} \neq \emptyset$ and $\mathcal{S}_l > 0$, provided (f_1) , (f_2) and (f_4') holds. In fact, this follow as in [16, Lemma 2.6 and Remark 2.8]. Indeed, let $v_R \in C_0^\infty(\mathbb{R})$, $R > 0$, such that $0 \leq v_R(t) \leq t_0$ and

$$v_R(t) = \begin{cases} t_0, & \text{if } |t| \leq R, \\ 0, & \text{if } |t| > R + 1. \end{cases}$$

For all $x \in \mathbb{R}^N$, taking $\varphi_R(x) := v_R(|x|)$, we have $\varphi_R \in \mathcal{D}^{s,2}(\mathbb{R}^N)$. Moreover,

$$\begin{aligned} \int_{\mathbb{R}^N} F(\varphi_R) dx &= \int_{B_R(x_0)} F(t_0) dx + \int_{B_{R+1}(x_0) \setminus B_R(x_0)} F(\varphi_R) dx \\ &\geq F(t_0)|B_R| - |B_{R+1} \setminus B_R| \left(\max_{t \in [0, t_0]} |F(t)| \right). \end{aligned}$$

Thus there exists two positive constant C_1 and C_2 such that

$$\int_{\mathbb{R}^N} F(\varphi_R) dx \geq C_1 R^N - C_2 R^{N-1} > 0,$$

provided that R is taken large enough. Taking a suitable $\sigma > 0$, we may conclude that $\Phi(\varphi_R(\cdot/\sigma)) = 1$.

Example 3.7. Typical examples (see Section 6 and the proof of Lemma 6.4) of a functions satisfying (f₁)–(f₈) are given by

(i) $f(x, t) = b(x)t|t|^{2_s^*-2}$, where $b(x) \in C(\mathbb{R}^N)$ with

$$b(x) > b(0) = \inf_{x \in \mathbb{R}^N} b(x) = \lim_{|x| \rightarrow \infty} b(x), \quad (3.6)$$

and $b(0) > 0$.

(ii) $f(x, t) = \exp\{b(x)(\sin(\ln |t|) + 2)\}(b(x) \cos(\ln |t|) + 2_s^*)|t|^{2_s^*-2}t$, with $f(x, 0) \equiv 0$; where $b(x) \in C(\mathbb{R}^N)$ satisfies (3.6), $b(0) = 0$ and moreover

$$\sup_{x \in \mathbb{R}^N} b(x) < 2_s^* - \sigma, \quad \text{for some } \sigma \in (2, 2_s^*).$$

The primitive is given by $F(x, t) = \exp\{b(x)(\sin(\ln |t|) + 2)\}|t|^{2_s^*}$.

4. PRELIMINARIES

4.1. Fractional Sobolev spaces. Let $0 < s < N/2$, by Placherel Theorem, we have

$$\|u\|^2 = \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 dx, \quad \text{for all } u \in C_0^\infty(\mathbb{R}^N).$$

Thus, by the well know inequality

$$\int_{\mathbb{R}^N} |u|^{2_s^*} dx \leq \mathcal{K}_* \left(\int_{\mathbb{R}^N} |\xi|^{2s} |\mathcal{F}u|^2 d\xi \right)^{2_s^*/2}, \quad \text{for all } u \in C_0^\infty(\mathbb{R}^N),$$

where

$$\mathcal{K}_* = \left[2^{-2s} \frac{\Gamma(\frac{N-2s}{2})}{\Gamma(\frac{N+2s}{2})} \left(\frac{\Gamma(N)}{\Gamma(N/2)} \right)^{2s/N} \right]^{2_s^*/2}$$

and $\mathcal{F}u$ is defined in (1.1), the space $\mathcal{D}^{s,2}(\mathbb{R}^N)$ is well defined with continuous embedding

$$\mathcal{D}^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N) \quad \text{for } 0 < s < N/2. \quad (4.1)$$

Moreover, $\mathcal{D}^{s,2}(\mathbb{R}^N)$ as a separable Hilbert space when endowed with the inner product

$$(u, v) = \int_{\mathbb{R}^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} v dx, \quad \text{for all } u, v \in \mathcal{D}^{s,2}(\mathbb{R}^N),$$

as well the characterization

$$\begin{aligned}\mathcal{D}^{s,2}(\mathbb{R}^N) &= \left\{ u \in L^{2^*}(\mathbb{R}^N) : (-\Delta)^{s/2}u \in L^2(\mathbb{R}^N) \right\} \\ &= \left\{ u \in L^{2^*}(\mathbb{R}^N) : |\xi|^s \mathcal{F}u \in L^2(\mathbb{R}^N) \right\}.\end{aligned}$$

For $\Omega \subset \mathbb{R}^N$ open set and $0 < s < 1$, the inhomogeneous fractional Sobolev space is defined as

$$H^s(\Omega) = \left\{ u \in L^2(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy < \infty \right\}, \quad (4.2)$$

with the norm

$$\|u\|_{H^s(\Omega)}^2 := \int_{\Omega} u^2 dx + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy.$$

When $0 < s < 1$, by [15, Proposition 3.4],

$$\|u\|^2 = \frac{C(N, s)}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy, \quad \forall u \in \mathcal{D}^{s,2}(\mathbb{R}^N),$$

for some positive constant $C(N, s)$. Thus, when $\Omega = \mathbb{R}^N$, we have

$$\begin{aligned}H^s(\mathbb{R}^N) &= \left\{ u \in L^2(\mathbb{R}^N) : |\xi|^s \mathcal{F}u \in L^2(\mathbb{R}^N) \right\} \\ &= \left\{ u \in L^2(\mathbb{R}^N) : (-\Delta)^{s/2}u \in L^2(\mathbb{R}^N) \right\}.\end{aligned} \quad (4.3)$$

Turns out that definition of $H^s(\mathbb{R}^N)$ given in (4.3) it is more appropriated for the general case $s \geq 0$, than definition (4.2), because for $s \geq 1$, the integral in (4.2) is finite if and only if u is constant (see [9, Proposition 2]). Moreover, we have the continuous embedding

$$H^s(\Omega) \hookrightarrow L^p(\Omega), \quad 2 \leq p \leq 2_s^*, \quad \text{for } 0 < s < N/2, \quad (4.4)$$

and the following compact embedding (see [15, Section 7]),

$$H^s(\mathbb{R}^N) \hookrightarrow L_{\text{loc}}^p(\mathbb{R}^N), \quad 1 \leq p < 2_s^*, \quad \text{for } 0 < s < 1. \quad (4.5)$$

Consequently, we have that every bounded sequence in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ has subsequence that converges almost everywhere and weakly in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, for $0 < s < \min\{1, N/2\}$. Let \mathcal{S}_0 the subspace of \mathcal{S} consisting in all function u such that $\mathcal{F}u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$. In this case, $(-\Delta)^s u \in \mathcal{S}$. We finish this section emphasizing that the Plancherel Theorem also gives the next identity, which it will be used several times throughout this paper

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2}u (-\Delta)^{s/2}v dx = \int_{\mathbb{R}^N} (-\Delta)^s uv dx, \quad \forall u, v \in \mathcal{S}_0. \quad (4.6)$$

4.2. Local regularity and Pohozaev Identity. We are in the position to prove that weak solutions of autonomous form of Eq. (\mathcal{P}_s) are $C^1(\mathbb{R}^N)$ and satisfies the Pohozaev identity

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 dx = \frac{2N}{N-2s} \int_{\mathbb{R}^N} F(u) dx, \quad (4.7)$$

under suitable assumptions on $f(t)$ (see Proposition 4.4 for the precise statement). We refer to [12], where the identity was studied for solutions in $H^s(\mathbb{R}^N)$ and when $f(t)$ satisfies a fractional version of the H. Berestycki and P.-L. Lions assumptions. The main idea for that, it is to use the so called Caffarelli-Silvestre extension (see [11] for more details) which transform the autonomous non-local Eq. (\mathcal{P}_s) in a local one and use recent regularity results to develop the resultant expression in a such way to apply the argument of [5, Section 2]. Our approach is in some way different from the usual one. Although we continue using Caffarelli-Silvestre extension

(also know as harmonic extension), by the results of [18] and [23], we can derive a local regularity for weak solutions in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ in a more suitable way to get the desired identity by applying a truncation argument. For bounded domains we refer to [34].

Next for the reader convenience we introduce the harmonic extension following [23, Section 2] and for that we begin describing a class of weight Sobolev spaces suitable to work with this harmonic extension. First, observe that, for any $0 < s < 1$, the function $z = (x, y) \mapsto |y|^{1-2s}$ belongs to the Muckenhoupt class \mathcal{A}_2 of weights in \mathbb{R}^{N+1} , that is

$$\left(\frac{1}{|B|} \int_B |y|^{1-2s} dx dy \right) \left(\frac{1}{|B|} \int_B |y|^{2s-1} dx dy \right) \leq C, \quad \forall \text{ ball } B \subset \mathbb{R}^{N+1}.$$

More details can be found in [17]. Let Q be a open set in \mathbb{R}^{N+1} , we consider $L^2(Q, |y|^{1-2s})$ as the Banach space of the Lebesgue measurable functions v defined in Q such that

$$\|v\|_{L^2(Q, |y|^{1-2s})} = \left(\int_Q |y|^{1-2s} v^2 dx dy \right)^{1/2} < \infty.$$

We also consider the space $H^1(Q, |y|^{1-2s})$ of the functions w in $L^2(Q, |y|^{1-2s})$ such that its weak derivatives w_{z_i} exists and belongs to $L^2(Q, |y|^{1-2s})$ for $i = 1, \dots, N+1$. It is easy to see that $H^1(Q, |y|^{1-2s})$ is a Hilbert space with inner product

$$(v_1, v_2)_{H^1(Q, |y|^{1-2s})} = \int_Q |y|^{1-2s} \langle \nabla v_1, \nabla v_2 \rangle + |y|^{1-2s} v_1 v_2 dx dy,$$

and the induced norm

$$\|v\|_{H^1(Q, |y|^{1-2s})} = \left(\int_Q |y|^{1-2s} |\nabla v|^2 + |y|^{1-2s} v^2 dx dy \right)^{1/2}.$$

We call attention to the fact that the space of smooth functions $C^\infty(Q) \cap H^1(Q, |y|^{1-2s})$ is dense in the weight Sobolev space $H^1(Q, |y|^{1-2s})$ (see [44] for further details).

Regarding the space $H^1(Q, y^{1-2s})$ with $Q = \Omega \times (0, R)$, where $\Omega \subset \mathbb{R}^N$ is a domain with Lipschitz boundary, it is well know the existence of a well-defined trace operator

$$t_r : H^1(Q, y^{1-2s}) \rightarrow H^s(\Omega)$$

with

$$\|t_r(v)\|_{H^s(\Omega)} \leq C \|v\|_{H^1(Q, y^{1-2s})}, \quad \forall v \in H^1(Q, y^{1-2s}),$$

where $C > 0$, depends only on N, s and Ω (see also [30]). Moreover, by the continuous embedding $H^s(\Omega) \hookrightarrow L^{2^*_s}(\Omega)$, we have

$$\|t_r(v)\|_{L^{2^*_s}(\Omega)} \leq C \|v\|_{H^1(Q, y^{1-2s})}, \quad \forall v \in H^1(Q, y^{1-2s}). \quad (4.8)$$

Let

$$P_s(x, y) = \beta(N, s) \frac{y^{2s}}{(|x|^2 + y^2)^{\frac{N+2s}{2}}},$$

where $\beta(N, s)$ is such that $\int_{\mathbb{R}^N} P_s(x, 1) dx = 1$ and $0 < s < \min\{1, N/2\}$. Considering the standard notation

$$\mathbb{R}_+^{N+1} = \{(x, y) \in \mathbb{R}^{N+1} : y > 0\},$$

for $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ let us set the s -harmonic extension of u ,

$$w(x, y) = E_s(u)(x, y) := \int_{\mathbb{R}^N} P_s(x - \xi, y) u(\xi) d\xi, \quad (x, y) \in \mathbb{R}_+^{N+1}.$$

Then, for any compact subset K of $\overline{\mathbb{R}_+^{N+1}}$, we have $w \in L^2(K, y^{1-2s})$, $\nabla w \in L^2(\mathbb{R}_+^{N+1}, y^{1-2s})$ and $w \in C^\infty(\mathbb{R}_+^{N+1})$. Moreover, w satisfies

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla w) = 0, & \text{in } \mathbb{R}_+^{N+1}, \\ -\lim_{y \rightarrow 0^+} y^{1-2s}w_y(x, y) = \kappa_s(-\Delta)^s u(x) & \text{in } \mathbb{R}^N, \\ \|\nabla w\|_{L^2(\mathbb{R}_+^{N+1}, y^{1-2s})}^2 = \kappa_s \|u\|^2, \end{cases} \quad (4.9)$$

where we understand (4.9) in the distribution sense, where $\kappa_s = 2^{1-2s}\Gamma(1-s)/\Gamma(s)$, and Γ is the gamma function. Precisely,

$$\int_{B_R^+} y^{1-2s} \langle \nabla w, \nabla \varphi \rangle \, dx dy = \kappa_s \int_{B_R^N} (-\Delta)^{s/2} u (-\Delta)^{s/2} (t_r \varphi) \, dx, \quad \forall \varphi \in C_0^\infty(B_R^+ \cup B_R^N),$$

where for $R > 0$,

$$\begin{cases} B_R = \{z = (x, y) \in \mathbb{R}^{N+1} : |z|^2 < R^2\}, \\ B_R^+ = B_R \cap \mathbb{R}_+^{N+1} \text{ and} \\ B_R^N = \{z = (x, y) \in \mathbb{R}_+^{N+1} : |z|^2 < R^2, y = 0\}. \end{cases}$$

More generally, given $g(t) \in C(\mathbb{R})$ such that

$$|g(t)| \leq C|t|^{2s^*-1}, \quad \forall t \in \mathbb{R}, \quad (4.10)$$

we say that a function $v \in H^1(B_R^+, y^{1-2s})$ is a weak solution of the problem

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla v) = 0 & \text{in } B_R^+, \\ -\lim_{y \rightarrow 0^+} y^{1-2s}v_y(x, y) = \kappa_s g(t_r(v)(x)) & \text{in } B_R^N, \end{cases} \quad (4.11)$$

if, for all $\varphi \in C_0^\infty(B_R^+ \cup B_R^N)$, we have

$$\int_{B_R^+} y^{1-2s} \langle \nabla v, \nabla \varphi \rangle \, dx dy = \kappa_s \int_{B_R^N} g(t_r(v)) t_r(\varphi) \, dx. \quad (4.12)$$

Thus $w = E_s(u)$ is a weak solution of (4.11) with $g(t) = f(t)$ if, and only if, u is a weak solution of Eq. (\mathcal{P}_s) .

Remark 4.1. Using the s -harmonic extension, it can be proved the existence of nonnegative weak solutions of (\mathcal{P}_s) if $f(x, t) \geq 0$ for all $t \geq 0$ and almost everywhere x in \mathbb{R}^N . For that one can consider the truncation

$$\bar{f}(x, t) = \begin{cases} f(x, t), & \text{if } t \geq 0 \\ 0, & \text{if } t < 0. \end{cases}$$

Thus for u a weak solution of (\mathcal{P}_s) , with $f(x, t)$ replaced by $\bar{f}(x, t)$, we have that u is also a weak solution for (\mathcal{P}_s) and is nonnegative. To see that, let $\xi \in C_0^\infty(\mathbb{R} : [0, 1])$ such that

$$\xi(t) = \begin{cases} 1, & \text{if } t \in [-1, 1] \\ 0, & \text{if } |t| \geq 2 \end{cases} \quad \text{and} \quad |\xi'(t)| \leq C \quad \text{for all } t \in \mathbb{R},$$

for some C positive constant. For each $n \in \mathbb{N}$, define $\xi_n : \mathbb{R}^{N+1} \rightarrow \mathbb{R}$ by $\xi_n(z) = \xi(|z|^2/n^2)$. Then $\xi_n \in C_0^\infty(\mathbb{R}^{N+1})$ and verifies

$$|\nabla \xi_n(z)| \leq C \quad \text{and} \quad |z| |\nabla \xi_n(z)| \leq C \quad \text{for all } z \in \mathbb{R}^{N+1}.$$

By a density argument, we can take $\varphi = \xi_n w_-$ in (4.12), where $w_-(z) = \min\{w(z), 0\}$. Since $w_-(z) = E_s(u_-)$, we have that

$$\begin{aligned} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} \xi_n |\nabla w_-|^2 + y^{1-2s} \xi_n \langle \nabla w_+, \nabla w_- \rangle + y^{1-2s} \langle \nabla w_+ + \nabla w_-, w_- \nabla \xi_n \rangle \, dx dy \\ = \kappa_s \int_{\mathbb{R}^N} \bar{f}(x, u) \xi_n u_- \, dx, \end{aligned}$$

and we may apply the Dominated Convergence Theorem and (4.9) to get

$$\|u_-\|^2 = \int_{\mathbb{R}^N} \bar{f}(x, u) u_- \, dx = 0.$$

Which implies that $u_- = 0$. On the other hand, if u has sufficient regularity one can show u is positive, by applying the maximum principle for the fractional Laplacian as described in [38].

In the first result of this section we derive, following the regularity results of [23], sufficient conditions over the harmonic extension which ensures the validity of identity (4.7).

Proposition 4.2. *Let $v \in H^1(B_R^+, y^{1-2s})$ be a weak solution of (4.11). Suppose that $g \in C^1(\mathbb{R})$ satisfies (4.10). If $t_r(v) \in L_{\text{loc}}^p(B_r^N)$, for some $p > 2_s^*$, then for any $R > 0$ there exists $0 < y_0, r < R$ with $B_r^N \times [0, y_0] \subset B_R^+$, and $\alpha \in (0, 1)$, such that*

$$v, \nabla_x v, y^{1-2s} v_y \in C^{0,\alpha}(B_r^N \times [0, y_0]), \quad (4.13)$$

where $\nabla_x v = (v_{x_1}, \dots, v_{x_N})$.

Proof. In fact, since

$$\frac{g(t_r v)}{1 + |t_r v|} \in L_{\text{loc}}^q(\mathbb{R}^N), \quad \forall N/2s < q < p/(2_s^* - 2),$$

we can see that (4.13) follows taking

$$g(t_r v) = \frac{g(t_r v)}{1 + |t_r v|} \operatorname{sgn}(t_r v) t_r v + \frac{g(t_r v)}{1 + |t_r v|},$$

and proceeding analogously to the proof of [23, Proposition 2.19], by applying [23, Proposition 2.6, Proposition 2.13, Theorem 2.14 and Lemma 2.18]. \square

In order to apply Proposition 4.2 we need to prove a Brezis-Kato type result (see [7]) for solutions of Eq. (\mathcal{P}_s). Although a similar result can be found in [18, Lemma 3.5], the absence of singularity in Eq. (\mathcal{P}_s) allows us to obtain a simpler proof. To achieve that, we strongly rely in the following lemmas, which enable us to proceed as in [7] (cf. [4, Proposition 5.1] or [45, Theorem 1.2]).

Lemma A. [17, Theorem 1.3] *For any $R > 0$, there exists $\sigma > 1$ and $C_R > 0$ depending on R , such that*

$$\left(\int_{B_R} |y|^{1-2s} |v|^{2\sigma} \, dx dy \right)^{1/\sigma} \leq C_R \int_{B_R} |y|^{1-2s} |\nabla v|^2 \, dx dy, \quad \forall v \in C_0^\infty(B_R).$$

Lemma B. [18, Lemma 2.6] *Let $\xi \in C(\mathbb{R}^{N+1})$ such that $\xi(z) = 0$ for all $|z| \geq R$. There exist $C > 0$ such that*

$$\left(\int_{B_R^N} |v \xi|^{2_s^*} \, dx dy \right)^{2/2_s^*} \leq C \int_{B_R^+} y^{1-2s} |\nabla(v \xi)|^2 \, dx dy, \quad \forall v \in H^1(B_R^+, y^{1-2s}).$$

Proposition 4.3. *Assume that condition (f₂) holds. Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a weak solution of Eq. (P_s) for the autonomous case, then $u \in L^p_{\text{loc}}(\mathbb{R}^N)$, for all $p \geq 1$.*

Proof. Let $w = E_s(u)$ and $\xi \in C_0^\infty(\mathbb{R}^{N+1} : [0, 1])$ such that

$$\xi(z) = \begin{cases} 1, & \text{if } |z| < R/2 \\ 0, & \text{if } |z| \geq R \end{cases} \quad \text{and} \quad |\nabla \xi(z)| \leq C \quad \forall z \in \mathbb{R}^{N+1},$$

for some $C > 0$. Since the map $t \mapsto t \min\{|t|^\beta, L\}$, $\beta, L > 0$, is Lipschitz in \mathbb{R} , considering $w_{\beta,L} := \min\{|w|^\beta, L\}$ we have $ww_{\beta,L} \in H^1(B_R^+, y^{1-2s})$, consequently using inequality (4.8) in a density argument one can see that $ww_{\beta,L}^2 \xi^2$ can be taken as a test function in definition (4.12). The main idea is to get the estimate

$$\int_{B_R^+} y^{1-2s} |\nabla(ww_{\beta,L} \xi)|^2 dx dy \leq C, \quad (4.14)$$

for a suitable β and $C > 0$ which does not depend on L . The next step is to use Fatou Lemma and Lemma B to obtain

$$\int_{B_R^N} |u|^{(\beta+1)2_s^*} dx \leq C.$$

This leads to a iteration procedure in β which implies in $u \in L^p(B_R^N)$ for all $p > 1$. To do so, we start taking

$$a(x) := \frac{|f(u)|}{1 + |u|} \in L_{\text{loc}}^{N/2s}(\mathbb{R}^N),$$

which implies

$$\int_{B_R^+} y^{1-2s} \langle \nabla w, \nabla(ww_{\beta,L}^2 \xi^2) \rangle dx dy \leq 2\kappa_s \int_{B_R^N} a(x)(1 + u^2) u_{\beta,L}^2 \xi^2 dx, \quad (4.15)$$

where we used that $(1+t)t \leq 2(1+t^2)$, $t > 0$ and $t_r(ww_{\beta,L}^2 \xi^2) = uu_{\beta,L}^2 \xi(\cdot, 0)^2$. We now compute the left side of the inequality (4.15) and use the following identity

$$w \langle \nabla w, \nabla(|w|^{2\beta}) \rangle = \frac{\beta}{2} |w|^{2(\beta-1)} |\nabla(w^2)|^2,$$

to conclude

$$\begin{aligned} & \int_{B_R^+} y^{1-2s} \min\{|w|^{2\beta}, L^2\} |\nabla w|^2 \xi^2 dx dy + \frac{\beta}{2} \int_{\{|w|^{2\beta} \leq L^2\} \cap B_R^+} y^{1-2s} |w|^{2(\beta-1)} |\nabla(w^2)|^2 \xi^2 dx dy \\ & \leq 2\kappa_s \int_{B_R^N} a(x)(1 + u^2) u_{\beta,L}^2 \xi^2 dx - 2 \int_{B_R^+} y^{1-2s} w \min\{|w|^{2\beta}, L^2\} \xi \langle \nabla w, \nabla \xi \rangle dx dy. \end{aligned} \quad (4.16)$$

Using the Cauchy inequality (with $\varepsilon = 1/4$) we have

$$\begin{aligned} & -2 \int_{B_R^+} y^{1-2s} w \min\{|w|^{2\beta}, L^2\} \xi \langle \nabla w, \nabla \xi \rangle dx dy \\ & \leq \frac{1}{2} \int_{B_R^+} y^{1-2s} \min\{|w|^{2\beta}, L^2\} |\nabla w|^2 \xi^2 dx dy + C \int_{B_R^+} y^{1-2s} w^2 \min\{|w|^{2\beta}, L^2\} |\nabla \xi|^2 dx dy, \end{aligned} \quad (4.17)$$

where $C > 0$ is independent of L . From replacing (4.17) in (4.16), we obtain

$$\begin{aligned} & \frac{1}{2} \int_{B_R^+} y^{1-2s} \min\{|w|^{2\beta}, L^2\} |\nabla w|^2 \xi^2 \, dx dy + \frac{\beta}{2} \int_{\{|w|^{2\beta} \leq L^2\} \cap B_R^+} y^{1-2s} |w|^{2(\beta-1)} |\nabla(w^2)|^2 \xi^2 \, dx dy \\ & \leq C \int_{B_R^+} y^{1-2s} w^2 \min\{|w|^{2\beta}, L^2\} |\nabla \xi|^2 \, dx dy + 2\kappa_s \int_{B_R^N} a(x)(1+u^2) u_{\beta,L}^2 \xi^2 \, dx. \end{aligned} \quad (4.18)$$

Now using

$$\beta^2 |w|^{2(\beta-1)} |\nabla(w^2)|^2 = 4w^2 \left| \nabla(|w|^\beta) \right|^2,$$

together with inequality (4.18), we can finally estimate (4.14),

$$\begin{aligned} & \int_{B_R^+} y^{1-2s} |\nabla(ww_{\beta,L}\xi)|^2 \, dx dy \\ & \leq C \int_{B_R^+} y^{1-2s} w^2 \min\{|w|^{2\beta}, L^2\} |\nabla \xi|^2 \, dx dy + 2\kappa_s \int_{B_R^N} a(x)(1+u^2) u_{\beta,L}^2 \xi^2 \, dx. \end{aligned} \quad (4.19)$$

It remains to estimate the last two terms in (4.19). Assuming $|u|^{\beta+1} \in L^2(B_R^N)$, we get

$$\begin{aligned} \int_{B_R^N} a(x) u^2 u_{\beta,L}^2 \xi^2 \, dx & \leq L_0 \int_{B_R^N} |u|^{2(\beta+1)} \xi^2 \, dx + \int_{B_R^N \cap \{a(x) \geq L_0\}} a(x) u^2 u_{\beta,L}^2 \xi^2 \, dx \\ & \leq C_1 L_0 + \tilde{C}_1 \varepsilon(L_0) \left(\int_{B_R^+} y^{1-2s} |\nabla(ww_{\beta,L}\xi)|^2 \, dx dy \right)^{2/2_s^*}, \end{aligned}$$

where

$$\varepsilon(L_0) := \left(\int_{\{a(x) \geq L_0\}} a^{N/2s}(x) \, dx \right)^{2s/N} \rightarrow 0, \text{ as } L_0 \rightarrow \infty.$$

By the same calculation and using $\min\{|t|^\beta, L\} \leq |t| \min\{|t|^\beta, L\} + 1, L > 1$, we obtain

$$\begin{aligned} & \int_{B_R^N} a(x) u_{\beta,L}^2 \xi^2 \, dx \\ & \leq C_2 L_0 + \tilde{C}_2 \varepsilon(L_0) \left[\left(\int_{B_R^+} y^{1-2s} |\nabla(ww_{\beta,L}\xi)|^2 \, dx dy \right)^{2/2_s^*} + \left(\int_{B_R^N} |\xi|^{2_s^*} \, dx \right)^{2/2_s^*} \right], \end{aligned}$$

Thus, we can take L_0 large enough such that

$$\int_{B_R^+} y^{1-2s} |\nabla(ww_{\beta,L}\xi)|^2 \, dx dy \leq C_3 \int_{B_R^+} y^{1-2s} w^2 \min\{|w|^{2\beta}, L^2\} |\nabla \xi|^2 \, dx dy.$$

Finally, assume that $\beta + 1 \leq \sigma$, where σ is given in Lemma A. Using the operator extension by reflection $\mathcal{R} : H^1(B_R^+, y^{1-2s}) \rightarrow H^1(B_R, |y|^{1-2s})$ given by

$$\mathcal{R}(w)(x, y) = \begin{cases} w(x, y), & \text{if } y > 0, \\ w(x, -y), & \text{if } y \leq 0, \end{cases}$$

(see for instance [11, Section 4]), we may apply Lemma A for an appropriated sequence of functions in $C_0^\infty(\mathbb{R}^{N+1})$, converging to $\mathcal{R}(w)$ in $H^1(B_R, |y|^{1-2s})$ to get

$$\int_{B_R^+} y^{1-2s} w^2 \min\{|w|^{2\beta}, L^2\} |\nabla \xi|^2 \, dx dy \leq C_4 \int_{B_R} |y|^{1-2s} |\nabla(\mathcal{R}(w))|^2 \, dx dy \leq C_5.$$

We take $\beta = \beta_1 = \min\{2_s^*/2, \sigma\} - 1$ and $\beta_{i+1} = \min\{2_s^*/2, \sigma\}(2_s^*/2)^i - 1$, $i = 0, 1, \dots$, to obtain that $u \in L_{\text{loc}}^{\beta_{i+1}}(\mathbb{R}^N)$. \square

Summing up all the previous results we can finally conclude the validity of identity (4.7) and the desired local regularity.

Proposition 4.4. *If $f(t) \in C^1(\mathbb{R})$ and satisfies (f_2) , then every weak solution of Eq. (\mathcal{P}_s) for the autonomous case belongs to $C^1(\mathbb{R}^N)$. Moreover, the Pohozaev identity (4.7) holds true.*

Proof. Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a weak solution of Eq. (\mathcal{P}_s) for the autonomous case with $f(t)$ satisfying (f_2) . Consider $w = E_s(u)$, then by Propositions 4.2, w possess the regularity (4.13). In particular, $\nabla u = \nabla w(x, 0) \in C(B_r^N)$ for any $r > 0$. Let $\xi \in C_0^\infty(\mathbb{R} : [0, 1])$ such that

$$\xi(t) = \begin{cases} 1, & \text{if } t \in [-1, 1] \\ 0, & \text{if } |t| \geq 2 \end{cases} \quad \text{and} \quad |\xi'(t)| \leq C \quad \forall t \in \mathbb{R},$$

for some $C > 0$. For each $n \in \mathbb{N}$, define $\xi_n : \mathbb{R}^{N+1} \rightarrow \mathbb{R}$ by $\xi_n(z) = \xi(|z|^2/n^2)$. Then $\xi_n \in C_0^\infty(\mathbb{R}^{N+1})$ and verifies

$$|\nabla \xi_n(z)| \leq C \quad \text{and} \quad |z| |\nabla \xi_n(z)| \leq C \quad \forall z \in \mathbb{R}^{N+1}, \quad (4.20)$$

for some $C > 0$. Now observe that,

$$\begin{aligned} \operatorname{div}(y^{1-2s} \nabla w) \langle z, \nabla w \rangle \xi_n &= \\ \operatorname{div} \left[y^{1-2s} \xi_n \left(\langle z, \nabla w \rangle \nabla w - \frac{|\nabla w|^2}{2} z \right) \right] &+ \frac{N-2s}{2} y^{1-2s} |\nabla w|^2 \xi_n \\ &+ y^{1-2s} \frac{|\nabla w|^2}{2} \langle z, \nabla \xi_n \rangle - y^{1-2s} \langle \nabla w, z \rangle \langle \nabla w, \nabla \xi_n \rangle. \end{aligned} \quad (4.21)$$

Given $\delta > 0$ we set

$$\begin{cases} B_{n,\delta} = \{z = (x, y) \in \mathbb{R}_+^{N+1} : |z|^2 < 2n^2, y > \delta\}, \\ F_{n,\delta}^1 = \{z = (x, y) \in \mathbb{R}_+^{N+1} : |z|^2 < 2n^2, y = \delta\}, \\ F_{n,\delta}^2 = \{z = (x, y) \in \mathbb{R}_+^{N+1} : |x|^2 + y^2 = 2n^2, y > \delta\}. \end{cases}$$

Hence $\partial B_{n,\delta} = F_{n,\delta}^1 \cup F_{n,\delta}^2$. Let $\eta(z) = (0, \dots, -1)$ be the unit outward normal vector of $B_{n,\delta}$ on $F_{n,\delta}^1$, since $\xi_n = 0$ on $F_{n,\delta}^2$, by condition (4.9), identity (4.21) and the Divergence Theorem we get

$$\begin{aligned} 0 &= \int_{B_{n,\delta}} \operatorname{div}(y^{1-2s} \nabla w) \langle z, \nabla w \rangle \xi_n \, dx dy \\ &= \int_{F_{n,\delta}^1} y^{1-2s} \xi_n \left[\langle z, \nabla w \rangle \langle \nabla w, \eta \rangle - \frac{|\nabla w|^2}{2} \langle z, \eta \rangle \right] \, dx dy + \vartheta_{n,\delta} \\ &= \int_{F_{n,\delta}^1} \xi_n \langle x, \nabla_x w \rangle (-y^{1-2s} w_y) \, dx - \int_{F_{n,\delta}^1} y^{1-2s} \xi_n |w_y|^2 y \, dx + \int_{F_{n,\delta}^1} y^{1-2s} \xi_n \frac{|\nabla w|^2}{2} y \, dx + \vartheta_{n,\delta} \\ &= I_{n,\delta}^1 + I_{n,\delta}^2 + I_{n,\delta}^3 + \vartheta_{n,\delta}, \end{aligned}$$

where

$$\vartheta_{n,\delta} = \int_{B_{n,\delta}} \frac{N-2s}{2} y^{1-2s} |\nabla w|^2 \xi_n + y^{1-2s} \frac{|\nabla w|^2}{2} \langle z, \nabla \xi_n \rangle - y^{1-2s} \langle \nabla w, z \rangle \langle \nabla w, \nabla \xi_n \rangle \, dx dy.$$

Using the same arguments as in [18, proof of Theorem 3.7] we deduce that there exists a sequence $\delta_k \rightarrow 0$ such that

$$I_{n,\delta_k}^2 + I_{n,\delta_k}^3 \rightarrow 0, \text{ as } k \rightarrow \infty.$$

Some computations leads to

$$\xi_n(x, 0) \langle x, \nabla u \rangle f(u) = \operatorname{div}(\xi_n(x, 0)F(u)x) - F(u) \langle \nabla \xi_n(x, 0), x \rangle - \xi_n(x, 0)F(u)N.$$

By condition (4.9) and the Divergence Theorem we have

$$\begin{aligned} \lim_{k \rightarrow \infty} I_{n,\delta_k}^1 &= \kappa_s \int_{B_{\sqrt{2}n}^N} \xi_n(x, 0) \langle x, \nabla u \rangle f(u) dx \\ &= \kappa_s \int_{B_{\sqrt{2}n}^N} \operatorname{div}(\xi_n(x, 0)F(u)x) - F(u) \langle \nabla \xi_n(x, 0), x \rangle - \xi_n(x, 0)F(u)N dx \\ &= -N\kappa_s \int_{B_{\sqrt{2}n}^N} \xi_n(x, 0)F(u)dx - \kappa_s \int_{B_{\sqrt{2}n}^N} F(u) \langle \nabla \xi_n(x, 0), x \rangle dx. \end{aligned}$$

Summing up, we have

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} (I_{n,\delta_k}^1 + I_{n,\delta_k}^2 + I_{n,\delta_k}^3 + \vartheta_{n,\delta_k}) \\ &= -N\kappa_s \int_{B_{\sqrt{2}n}^N} \xi_n F(u) dx - \kappa_s \int_{B_{\sqrt{2}n}^N} F(u) \langle \nabla \xi_n, x \rangle dx \\ &\quad + \int_{B_{\sqrt{2}n}^+} \frac{N-2s}{2} y^{1-2s} |\nabla w|^2 \xi_n + y^{1-2s} \frac{|\nabla w|^2}{2} \langle z, \nabla \xi_n \rangle - y^{1-2s} \langle \nabla w, z \rangle \langle \nabla w, \nabla \xi_n \rangle dx dy. \end{aligned}$$

Consequently taking $n \rightarrow \infty$ and using conditions (4.20), we conclude

$$\frac{N-2s}{2} \int_{\mathbb{R}_+^{N+1}} y^{1-2s} |\nabla w|^2 dx dy = N\kappa_s \int_{\mathbb{R}^N} F(u) dx, \quad (4.22)$$

which together with condition (4.9) implies (4.7), and the proof is complete. \square

4.3. D-weak convergence and dislocation spaces. As already mentioned, to achieve the decomposition described in Theorem 2.1, we follow the abstract approach of D -weak convergence and dislocation spaces developed in [43]. For the convenience of the reader we state the basic concepts without proofs, thus making our exposition self-contained. In this subsection H denotes a Hilbert space.

Definition C. [43, Definition 3.1] Let D be a set of bounded linear operators on H , such that for every $g \in D$, $\inf_{u \in H, \|u\|=1} \|gu\| > 0$. We will say that the sequence $(u_k) \subset H$ converges to u D -weakly in H , which we will denote as

$$u_k \xrightarrow{D} u, \text{ in } H,$$

if for any sequence $(g_k) \subset D$,

$$(g_k^* g_k)^{-1} g_k^* (u_k - u) \rightharpoonup 0 \text{ in } H.$$

For (g_k) a sequence of bounded linear operators in H , we use the notation $g_k \rightharpoonup 0$ to indicate that $g_k u \rightharpoonup 0$ in H for all $u \in H$.

Definition D. [43, Definition 3.2] Let H be a separable infinite-dimensional Hilbert space. A set D of bounded linear operators on H is a set of dislocations if

$$0 < \delta := \inf_{g \in D, \|u\|=1} \|gu\|^2 \leq \sup_{g \in D, \|u\|=1} \|gu\|^2 < \infty,$$

$$(u_k) \subset H, (g_k) \subset D, u_k \rightharpoonup 0 \text{ in } H \Rightarrow g_k^* g_k u_k \rightharpoonup 0 \text{ in } H,$$

and, whenever $(u_k) \subset H$ and $(g_k), (h_k) \subset D$,

$$h_k^* g_k \not\rightharpoonup 0, (g_k^* g_k)^{-1} g_k^* u_k \rightharpoonup 0 \text{ in } H \Rightarrow (h_k^* h_k)^{-1} h_k^* u_k \rightharpoonup 0 \text{ in } H.$$

The pair (H, D) is called a dislocation space.

The next result give a sufficient condition to establish if a pair (H, D) is a dislocation space.

Proposition E. [43, Proposition 3.1] Let H be a separable infinite-dimensional Hilbert space and D be a group (under the operator multiplication) of unitary operators $g : H \rightarrow H$, that is, $g^* = g^{-1}$. If

$$g_k \not\rightharpoonup 0 \text{ in } H, g_k \in D \Rightarrow g_k u \text{ has a convergent subsequence, for all } u \in H,$$

then (H, D) is dislocation space.

The next result provides a profile decomposition for bounded sequence in a suitable abstract Hilbert space, it is crucial to obtain the decomposition in Theorem (2.1), and it can be seen as a generalization of the celebrated Banach-Alaoglu-Bourbaki Theorem.

Theorem F. [43, Theorem 3.1] Let (H, D) be a dislocation space. If $(u_k) \subset H$ is a bounded sequence, then there exists a set $\mathbb{N}_0 \subset \mathbb{N}$, and sequences $(w^{(n)})_{n \in \mathbb{N}_0} \subset H$, $(g_k^{(n)})_{k \in \mathbb{N}} \subset D$, $g_k^{(1)} = \text{id}$, with $n \in \mathbb{N}_0$, such that for a subsequence of (u_k) ,

$$\begin{aligned} & \left(g_k^{(n)*} g_k^{(n)} \right)^{-1} g_k^{(n)*} u_k \rightharpoonup w^{(n)} \text{ in } H, \\ & g_k^{(n)*} g_k^{(m)} \rightharpoonup 0 \text{ for } n \neq m. \\ & \sum_{n \in \mathbb{N}_0} \|w^{(n)}\|^2 \leq \delta^{-1} \limsup_k \|u_k\|^2. \\ & u_k - \sum_{n \in \mathbb{N}_0} g_k^{(n)} w^{(n)} \xrightarrow{D} 0, \end{aligned}$$

where the series $\sum_{n \in \mathbb{N}_0} g_k^{(n)} w^{(n)}$ converges uniformly in k .

5. PROOF OF THEOREM 2.1

To prove it, roughly speaking, we take D as the group of dilations and translations (the precise description of D is given below) in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, and describe the behavior of those operators under the weak convergence. We consider

$$D_{\mathbb{R}^N} := \{g_y : \mathcal{D}^{s,2}(\mathbb{R}^N) \rightarrow \mathcal{D}^{s,2}(\mathbb{R}^N) : g_y u(x) = u(x - y), y \in \mathbb{R}^N\},$$

and for $\gamma > 1$,

$$\delta_{\mathbb{R}} := \left\{ \delta_j : \mathcal{D}^{s,2}(\mathbb{R}^N) \rightarrow \mathcal{D}^{s,2}(\mathbb{R}^N) : \delta_j u(x) = \gamma^{\frac{N-2s}{2}j} u(\gamma^j x), j \in \mathbb{R} \right\}, \quad (5.1)$$

the groups of operators on $\mathcal{D}^{s,2}(\mathbb{R}^N)$ induced by translations and dilations on \mathbb{R}^N , respectively. One can easily check that $D_{\mathbb{R}^N}$ and $\delta_{\mathbb{R}}$ are indeed groups of unitary operators in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, by using the following identities

$$\begin{cases} (-\Delta)^{s/2} (u(\cdot - y)) = ((-\Delta)^{s/2} u)(\cdot - y), \\ (-\Delta)^{s/2} (u(\tau \cdot)) = \tau^s ((-\Delta)^{s/2} u)(\tau \cdot), \end{cases} \quad (5.2)$$

for $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$, $y \in \mathbb{R}^N$ and $\tau > 0$. Now, we define the group

$$D_{\mathbb{R}^N, \mathbb{R}} := \left\{ d_{y,j} : \mathcal{D}^{s,2}(\mathbb{R}^N) \rightarrow \mathcal{D}^{s,2}(\mathbb{R}^N) : d_{y,j} u(x) := \gamma^{\frac{N-2s}{2}j} u(\gamma^j(x-y)), y \in \mathbb{R}^N, j \in \mathbb{R} \right\},$$

which consists by the composition of the elements of $D_{\mathbb{R}^N}$ with $\delta_{\mathbb{R}}$, i.e., $d_{y,j} = \delta_j \circ g_{\gamma^j y}$. By checking that $d_{y,j} \circ d_{z,l} = d_{y+\gamma^{-j}z, j+l}$ and $(d_{y,j})^{-1} = d_{-\gamma^j y, -j}$, it is easy to see that $D_{\mathbb{R}^N, \mathbb{R}}$ is a group of unitary operators in $\mathcal{D}^{s,2}(\mathbb{R}^N)$.

With the preceding notation we first derive the next basic result.

Lemma 5.1. *Let $(y_k, j_k) \subset \mathbb{R}^N \times \mathbb{R}$, such that $(y_k, j_k) \rightarrow (y, j)$. Then*

$$d_{y_k, j_k} u \rightarrow d_{y, j} u, \quad \forall u \in \mathcal{D}^{s,2}(\mathbb{R}^N).$$

Proof. By the density of \mathcal{S}_0 in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, we just need to prove in the case where $u \in \mathcal{S}_0$. Consequently,

$$\|d_{y_k, j_k} u - d_{y, j} u\|^2 = 2\|u\|^2 - 2\gamma^{\frac{N}{2}(j_k + j)} \int_{\mathbb{R}^N} (-\Delta)^{s/2} u(\gamma^{j_k}(x - y_k)) (-\Delta)^{s/2} u(\gamma^j(x - y)) dx,$$

which together with (4.6), implies

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} u(\gamma^{j_k}(x - y_k)) (-\Delta)^{s/2} u(\gamma^j(x - y)) dx = \int_{\mathbb{R}^N} u(\gamma^{j_k}(x - y_k)) (-\Delta)^s u(\gamma^j(x - y)) dx.$$

Since $(-\Delta)^s u(\gamma^j(\cdot - y)) \in L^1(\mathbb{R}^N)$ and $|u(\gamma^{j_k}(x - y_k))| \leq \|u\|_{L^\infty(\mathbb{R}^N)}$ a.e. in \mathbb{R}^N , the assertion follows by the Dominated Convergence Theorem. \square

We shall describe how the elements of $D_{\mathbb{R}^N, \mathbb{R}}$ acts in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. This is done in the next result, which is a slight different version of [31, Lemma 3].

Lemma 5.2. *Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N) \setminus \{0\}$. The sequence $(d_{y_k, j_k} u)$, with $(y_k, j_k) \subset \mathbb{R}^N \times \mathbb{R}$, converges weakly to zero if and only if $|j_k| + |y_k| \rightarrow \infty$.*

Proof. Suppose first that $d_{y_k, j_k} u \rightharpoonup 0$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ and assume, by contradiction, that we have $|j_k| + |y_k| \not\rightarrow \infty$. Then, up to subsequences, we may assume that $y_k \rightarrow y \in \mathbb{R}^N$ and $j_k \rightarrow j \in \mathbb{R}$, as $k \rightarrow \infty$. By Lemma 5.1,

$$0 = \lim_{k \rightarrow \infty} (d_{y_k, j_k} u, d_{y, j} u) = \|d_{y, j} u\|^2 = \|u\|^2,$$

a contradiction with the fact that $u \neq 0$.

Conversely, assume that $|j_k| + |y_k| \rightarrow \infty$. By density of \mathcal{S}_0 in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ it suffices to prove

$$(d_{y_k, j_k} u, v) \rightarrow 0, \quad \forall u, v \in \mathcal{S}_0.$$

If we prove that every subsequence of $(d_{y_k, j_k} u)$ has a subsequence that weakly converges to zero, the assertion follows. To do this, we divide the proof in two cases:

- (i) There exists a subsequence of (j_k) , such that $j_k \rightarrow +\infty$ or $-\infty$;
- (ii) There exists a convergent subsequence of (j_k) , such that $j_k \rightarrow j_0$ and $|y_k| \rightarrow \infty$.

Before we start analyzing each case, we observe that by using identity (4.6), one has

$$(d_{y_k, j_k} u, v) = \int_{\mathbb{R}^N} (-\Delta)^s v d_{y_k, j_k} u \, dx. \quad (5.3)$$

Therefore it is sufficient to study the desired convergence in the right-hand side of (5.3).

Case (i). Assume first that $j_k \rightarrow +\infty$. By changing the variables under the integral we have that

$$\begin{aligned} |(d_{y_k, j_k} u, v)| &= \gamma^{\frac{N-2s}{2} j_k} \left| \int_{\mathbb{R}^N} (-\Delta)^s v(x) u(\gamma^{j_k}(x - y_k)) \, dx \right| \\ &\leq \gamma^{-\frac{N+2s}{2} j_k} \|(-\Delta)^s v\|_{L^\infty(\mathbb{R}^N)} \|u\|_{L^1(\mathbb{R}^N)} \rightarrow 0, \text{ as } k \rightarrow \infty. \end{aligned}$$

The same conclusion holds when $j_k \rightarrow -\infty$. Indeed, since $D_{\mathbb{R}^N, \mathbb{R}}$ is a group

$$(d_{y_k, j_k} u, v) = (u, (d_{y_k, j_k})^{-1} v) = (u, d_{-\gamma^{j_k} y_k, -j_k} v).$$

Hence, by interchanging u and v we get the desired conclusion.

Case (ii). Since $j_k \rightarrow j_0$, we have $d_{y_k, j_k} u(x) \rightarrow 0$ a.e. in \mathbb{R}^N . Also,

$$|d_{y_k, j_k} u(x) (-\Delta)^s v(x)| \leq C \|u\|_{L^\infty(\mathbb{R}^N)} |(-\Delta)^s v(x)|, \text{ a.e. in } \mathbb{R}^N,$$

where C is a positive constant which do not depends on k . Thus, by the Dominated Convergence Theorem,

$$(d_{y_k, j_k} u, v) \rightarrow 0,$$

which completes the proof. \square

Finally we take $D = D_{\mathbb{Z}^N, \mathbb{Z}} = \{d_{y, j} \in D_{\mathbb{R}^N, \mathbb{R}} : y \in \mathbb{Z}^N, j \in \mathbb{Z}\}$, as the aforementioned group of unity operators in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. The main reason for this (instead of $D_{\mathbb{R}^N, \mathbb{R}}$) in one of the statements in Theorem 2.1: it gives further properties for the weak decomposition (cf. Theorem F or [31, Theorem 8]). Considering the following cocompactness result we are able to prove Theorem 2.1 (a similar result can be found in [31, Proposition 1]).

Proposition 5.3. *Let (u_k) be a bounded sequence in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Then $u_k \xrightarrow{D} 0$ if and only if $u_k \rightarrow 0$ in $L^{2^*}_s(\mathbb{R}^N)$.*

Proof. Our proof follows the same ideas of [43, Lemma 5.3]. Since $C_0^\infty(\mathbb{R}^N)$ is a dense subset of $\mathcal{D}^{s,2}(\mathbb{R}^N)$, by the continuous embedding of $\mathcal{D}^{s,2}(\mathbb{R}^N)$ in $L^{2^*}_s(\mathbb{R}^N)$, we can assume without loss of generality that the sequence (u_k) belongs to $C_0^\infty(\mathbb{R}^N)$. Let us suppose first that $u_k \xrightarrow{D} 0$. Consider $\xi \in C_0^\infty(\mathbb{R}, [0, \infty))$ such that

$$\xi(t) = \begin{cases} t, & \text{if } \frac{1}{4} \gamma^{\frac{N-2s}{2}} \leq t \leq \frac{3}{4} \gamma^{\frac{N-2s}{2}}, \\ 0, & \text{if } t \leq 1 \text{ or } t \geq \gamma^{\frac{N-2s}{2}}, \end{cases} \quad \text{and } |\xi'(t)| \leq C, \forall t,$$

where we can assume without loss of generality that $\gamma > 4$, because we can replace it by $\gamma^{n_0} > 4$, for integer n_0 large enough, if necessary. Notice that there exists a positive constant C such that

$$\begin{cases} |\xi(t)|^{2^*}_s \leq C t^2, \\ |\xi(t)|^2 \leq C |t|^{2^*}_s, \end{cases} \quad \forall t \in \mathbb{R}. \quad (5.4)$$

Given any sequence (j_k) in \mathbb{Z} , denote

$$v_k(x) = \gamma^{\frac{N-2s}{2} j_k} u_k(\gamma^{j_k} x).$$

Let $Q_z = (0, 1)^N + z$, with $z \in \mathbb{Z}^N$. By the Sobolev embedding (4.5), for any $z \in \mathbb{Z}^N$, we get that

$$\int_{Q_z} |\xi(|v_k|)|^{2_s^*} dx \leq C \|\xi(|v_k|)\|_{H^s(Q_z)}^2 \left(\int_{Q_z} v_k^2 dx \right)^{1-2/2_s^*}. \quad (5.5)$$

Moreover, embedding (4.4) and relations (5.4) implies that,

$$\begin{aligned} \sum_{z \in \mathbb{Z}} \|\xi(|v_k|)\|_{H^s(Q_z)}^2 &= \sum_{z \in \mathbb{Z}} \int_{Q_z} |\xi(|v_k|)|^2 dx + \int_{Q_z} \int_{Q_z} \frac{|\xi(|v_k|)(x) - \xi(|v_k|)(y)|^2}{|x - y|^{N+2s}} dx dy \\ &\leq \int_{\mathbb{R}^N} |\xi(|v_k|)|^2 dx + \max_{t \geq 0} \xi'(t) \sum_{z \in \mathbb{Z}} \int_{Q_z} \int_{Q_z} \frac{|v_k(x) - v_k(y)|^2}{|x - y|^{N+2s}} dx dy \leq C \|v_k\|^2. \end{aligned}$$

Thus, we can take the sum over $z \in \mathbb{Z}^N$ in (5.5) to obtain

$$\int_{\mathbb{R}^N} |\xi(|v_k|)|^{2_s^*} dx \leq C \sup_{z \in \mathbb{Z}^N} \left(\int_{Q_z} v_k^2 dx \right)^{1-2/2_s^*}. \quad (5.6)$$

For each k , let $y_k \in \mathbb{Z}^N$ such that

$$\sup_{z \in \mathbb{Z}^N} \left(\int_{Q_z} v_k^2 dx \right)^{1-2/2_s^*} \leq 2 \left(\int_{Q_{y_k}} v_k^2 dx \right)^{1-2/2_s^*}. \quad (5.7)$$

Since $u_k \xrightarrow{D} 0$, we have that $v_k(\cdot - z_k) \rightarrow 0$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, which allow us to apply embedding (4.5) and obtain that

$$\int_{Q_{z_k}} v_k^2 dx = \int_{(0,1)^N} v_k^2(\cdot - z_k) dx \rightarrow 0, \text{ as } k \rightarrow \infty. \quad (5.8)$$

Replacing (5.7) and (5.8) in (5.6) we conclude that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} |\xi(|v_k|)|^{2_s^*} dx = 0. \quad (5.9)$$

Now let

$$\xi_j(t) = \gamma^{-\frac{N-2s}{2}j} \xi(\gamma^{\frac{N-2s}{2}j} t), \quad j \in \mathbb{Z}.$$

From convergence (5.9), we get

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} |\xi_{j_k}(|u_k|)|^{2_s^*} dx = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} |\xi(|v_k|)|^{2_s^*} dx = 0, \quad \text{for any sequence } (j_k) \text{ in } \mathbb{Z}. \quad (5.10)$$

Now the embedding $\mathcal{D}^{s,2}(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ enable us to get the following estimate,

$$\int_{\mathbb{R}^N} |\xi_j(|u_k|)|^{2_s^*} dx \leq C \|\xi_j(|u_k|)\|^2 \left(\int_{\mathbb{R}^N} |\xi_j(|u_k|)|^{2_s^*} dx \right)^{1-2/2_s^*}. \quad (5.11)$$

For $j \in \mathbb{Z}$, let

$$\begin{cases} D_{j,k} = \left\{ x \in \mathbb{R}^N : \gamma^{-\frac{N-2s}{2}j} \leq |u_k(x)| < \gamma^{-\frac{N-2s}{2}(j-1)} \right\}; \\ E_{j,k} = (D_{j,k} \times \mathbb{R}^N) \cup (\mathbb{R}^N \times D_{j,k}); \\ L_{j,k} = \left\{ x \in \mathbb{R}^N : \frac{1}{4} \gamma^{-\frac{N-2s}{2}j} \leq |u_k(x)| \leq \frac{3}{4} \gamma^{-\frac{N-2s}{2}(j-1)} \right\}, \end{cases} \quad (5.12)$$

Since u_k is smooth and has compact support, there exists j_0 in \mathbb{Z} and l in \mathbb{N} such that

$$\text{supp}(u_k) \subset \bigcup_{j=0}^l L_{j+j_0,k} \subset \bigcup_{j=0}^l D_{j+j_0,k},$$

We also have that the sets

$$S_{j,k} = \bigcup_{m=0}^j E_{j+j_0,k} \cap E_{m+j_0,k}, \quad j = 1, \dots, l,$$

are disjunct as well $E_{j_0,k}$ and $E_{j+j_0,k} \setminus S_{j,k}$, for $j = 1, \dots, l$. Thus we may write

$$\begin{aligned} \sum_{j=0}^l \iint_{E_{j+j_0,k}} \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{N+2s}} dx dy &= \sum_{j=1}^l \iint_{S_{j,k}} \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{N+2s}} dx dy \\ &\quad + \iint_{E_{j_0,k}} \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{N+2s}} dx dy + \sum_{j=1}^l \iint_{E_{j+j_0,k} \setminus S_{j,k}} \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{N+2s}} dx dy, \\ &= \iint_{A_{l,k}} \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{N+2s}} dx dy + \iint_{B_{l,k}} \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{N+2s}} dx dy, \end{aligned}$$

where

$$A_{l,k} = E_{j_0,k} \cup \bigcup_{j=1}^l E_{j+j_0,k} \setminus S_{j,k} \quad \text{and} \quad B_{l,k} = \bigcup_{j=1}^l S_{j,k},$$

to get that the estimate

$$\begin{aligned} \sum_{j=0}^l \|\xi_j(|u_k|)\|^2 &= \frac{C(N,s)}{2} \sum_{j=0}^l \iint_{E_{j,k}} \frac{|\xi_j(|u_k|)(x) - \xi_j(|u_k|)(y)|^2}{|x - y|^{N+2s}} dx dy \\ &\leq \frac{C(N,s)}{2} \max_{t \geq 0} \xi'(t) \sum_{j=0}^l \iint_{E_{j,k}} \frac{|u_k(x) - u_k(y)|^2}{|x - y|^{N+2s}} dx dy \leq 2 \max_{t \geq 0} \xi'(t) \|u_k\|^2. \end{aligned}$$

Moreover,

$$\begin{aligned} \int_{\mathbb{R}^N} |u_k|^{2_s^*} dx &\leq \sum_{j=0}^l \int_{L_{j,k}} |u_k|^{2_s^*} dx \\ &\leq \sum_{j=0}^l \int_{L_{j,k}} |u_k|^{2_s^*} dx + \int_{D_{j,k} \setminus L_{j,k}} |\xi_j(|u_k|)|^{2_s^*} dx = \sum_{j=0}^l \int_{\mathbb{R}^N} |\xi_j(|u_k|)|^{2_s^*} dx. \end{aligned}$$

In view of that, we take the sum over $j = 0, \dots, l$ in (5.11) to conclude that

$$\int_{\mathbb{R}^N} |u_k|^{2_s^*} dx \leq C \sup_{j \in \mathbb{Z}} \left(\int_{\mathbb{R}^N} |\xi_j(|u_k|)|^{2_s^*} dx \right)^{1-2/2_s^*}.$$

Similarly as before, we choose (j_k) such that

$$\sup_{j \in \mathbb{Z}} \left(\int_{\mathbb{R}^N} |\xi_j(|u_k|)|^{2_s^*} dx \right)^{1-2/2_s^*} \leq 2 \left(\int_{\mathbb{R}^N} |\xi_{j_k}(|u_k|)|^{2_s^*} dx \right)^{1-2/2_s^*},$$

which, from (5.10) implies that $|u_k|_{2_s^*} \rightarrow 0$.

Now assume that $u_k \rightarrow 0$ in $L^{2_s^*}(\mathbb{R}^N)$. Let us argue by contradiction and suppose that there exists (y_k) in \mathbb{Z}^N and (j_k) in \mathbb{Z} such that $d_{y_k, j_k} u_k \rightharpoonup u \neq 0$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. The invariance of d_{y_k, j_k} with respect to the $L^{2_s^*}$ norm leads to

$$\|u\|_{L_s^*(\mathbb{R}^N)} \leq \liminf_{k \rightarrow \infty} \|d_{y_k, j_k} u_k\|_{L_s^*(\mathbb{R}^N)} = \lim_{k \rightarrow \infty} \|u_k\|_{L^\infty(\mathbb{R}^N)} = 0,$$

which is a contradiction with the fact that $u \neq 0$. \square

Proof of Theorem 2.1 completed. By Theorem F, we first need to prove that $(\mathcal{D}^{s,2}(\mathbb{R}^N), D_{\mathbb{Z}^N, \mathbb{Z}})$ is a dislocation space. To do so, we use Proposition E. Let $(d_{y_k, j_k}) \subset D_{\mathbb{Z}^N, \mathbb{Z}}$, such that $d_{y_k, j_k} \not\rightarrow 0$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Hence by Lemma 5.2, $y_k \rightarrow y$ and $j_k \rightarrow j$, up to a subsequence, and by Lemma 5.1, $d_{y_k, j_k} u \rightarrow d_{y, j} u$, for all $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$. Therefore Theorem F holds with $H = \mathcal{D}^{s,2}(\mathbb{R}^N)$ and $D = D_{\mathbb{Z}^N, \mathbb{Z}}$. It follows immediately assertions (2.1) and (2.3). The assertion (2.2) is guaranteed by Lemma 5.2, and (2.3) follows from Proposition 5.3. Finally, for each $n \in \mathbb{N}_*$, if $(j_k^{(n)})$ is unbounded we can replace it by a subsequence convergent to $+\infty$ or $-\infty$, by checking either $\limsup_k j_k^{(n)} = +\infty$ or $\limsup_k j_k^{(n)} = -\infty$. If $(j_k^{(n)})$ is bounded, we can replace it by a constant subsequence, say $j^{(n)}$. Moreover, by taking $v_k^{(n)}(x) = \gamma^{-\frac{N-2s}{2}j^{(n)}} u_k(\gamma^{-j^{(n)}}x + y_k^{(n)})$, the convergence (2.1) implies

$$u_k(\cdot + y_k^{(n)}) = \delta_{-j^{(n)}} v_k^{(n)} \rightharpoonup \delta_{-j^{(n)}} w^{(n)} \text{ in } \mathcal{D}^{s,2}(\mathbb{R}^N),$$

thus we may set $j^{(n)} = 0$ and rename $\delta_{-j^{(n)}} w^{(n)}$ as $w^{(n)}$. Since \mathbb{N}_* is possibly infinite, the conclusion follows by a standard diagonal argument in the extraction of each successive subsequence. \square

6. SELF-SIMILAR FUNCTIONS

We now pass to study a class of non-linearity consistent with our profile decomposition. As it can be seen in the following examples, this class of nonlinearity can be seen as asymptotically oscillatory about the critical power $|t|^{2_s^*}$.

Definition 6.1. We say that $F \in C(\mathbb{R})$ is fractional self-similar if there exist $\gamma > 1$ and $0 < s < \min\{1, N/2\}$ such that

$$F(t) = \gamma^{-Nj} F(\gamma^{\frac{N-2s}{2}j} t), \quad \forall j \in \mathbb{Z}, t \in \mathbb{R}.$$

In this case we use to say that F is fractional self-similar with factor γ and fraction s .

Example 6.2. Typical examples of self-similar functions are

- (i) $F(t) = |t|^{2_s^*}$, which is self-similar for every factor γ and fraction $0 < s < \min\{1, N/2\}$;
- (ii) $H(t) = \cos(\ln |t|)|t|^{2_s^*}$, $H(0) := 0$, which is self-similar with factor $e^{4\pi/(N-2s)}$ and every fraction $0 < s < \min\{1, N/2\}$.

Remark 6.3. The function $F(t) \in C^1(\mathbb{R})$ is self-similar if, and only if

$$F'(t) = \gamma^{-\frac{N+2s}{2}j} F'(\gamma^{\frac{N-2s}{2}j} t), \quad \forall j \in \mathbb{Z}, \text{ and } t \in \mathbb{R}.$$

For the local case a class of self-similar function was introduced in [41–43].

In the next result we derive the basic properties of self-similar functions.

Lemma 6.4. Assume that $F(t)$ is self-similar.

- (i) For each $u \in L^{2_s^*}(\mathbb{R}^N)$ and $j \in \mathbb{Z}$, we have

$$\int_{\mathbb{R}^N} F\left(\gamma^{\frac{N-2s}{2}j} u(\gamma^j x)\right) dx = \int_{\mathbb{R}^N} F(u) dx; \quad (6.1)$$

- (ii) There exists $C > 0$ such that

$$|F(t)| \leq C|t|^{2_s^*}, \quad \forall t \in \mathbb{R}. \quad (6.2)$$

Moreover, if $F \in C^2(\mathbb{R})$, then there exists $C > 0$, such that

$$|F(t)| + |F'(t)t| + |F''(t)t^2| \leq C|t|^{2_s^*}, \quad \forall t \in \mathbb{R}; \quad (6.3)$$

(iii) If $F(t)$ is locally Lipschitz then $F(t)$ satisfies (f₅).

Proof. (i) The identity (6.1) follows immediately by using the change of variables theorem in the integral on the left side of the equation.

(ii) Fix the interval $L = [\gamma^{-\frac{N-2s}{2}}, \gamma^{\frac{N-2s}{2}}]$. By continuity, there exists $C = C(L)$ such that $|F(t)| \leq Ct^{2s^*}$, for all $t \in L$. Now, let $0 < t < \gamma^{-\frac{N-2s}{2}}$ or $t > \gamma^{\frac{N-2s}{2}}$, then (in any case) there exists $j \in \mathbb{Z}$ such that $\gamma^{\frac{N-2s}{2}j}t \in L$, and consequently,

$$\gamma^{Nj}|F(t)| = |F(\gamma^{\frac{N-2s}{2}j}t)| \leq \gamma^{Nj}Ct^{2s^*}.$$

The case where $t < 0$ is analogous. The proof of (6.3) follow a similar argument.

(iii) The proof is by induction in M . So we just need to prove that there exists $C > 0$ such that

$$|F(a_1 + a_2) - F(a_1) - F(a_2)| \leq C \left(|a_1||a_2|^{2s^*-1} + |a_1|^{2s^*-1}|a_2| \right). \quad (6.4)$$

To do so, we first fix the interval $I = [-\gamma^{\frac{N-2s}{2}k}, \gamma^{\frac{N-2s}{2}k}]$, where $k \in \mathbb{Z}$ is taken such that $\gamma^{\frac{N-2s}{2}(k-1)} > 2$, to use the Lipschitz assumption. The proof follows by considering several cases.

Case 1: Suppose that $|a_1| \leq 1 \leq |a_2|$ and $a_1 + a_2 \in I$. Thus there exists $C = C(I)$ such that

$$|F(a_1 + a_2) - F(a_1) - F(a_2)| \leq C(|a_1| + |F(a_1)|).$$

By condition (6.2) we can estimate

$$|a_1| + |F(a_1)| \leq C(|a_1||a_2|^{2s^*-1} + |a_1|^{2s^*-1}|a_2|).$$

Case 2: Assume that $|a_1|, |a_2| \geq 1$ and $a_1 + a_2 \in I$. Then, there exists $j_1 \in \mathbb{Z}$, $j_1 \leq 0$, such that $|b_1| \leq 1$, where $b_1 := \gamma^{\frac{N-2s}{2}j_1}a_1$. It is easy to see that $b_1 + a_2 \in I$, hence by the first case, we have the following estimate

$$\begin{aligned} |F(b_1 + a_2) - F(b_1) - F(a_2)| &\leq \gamma^{\frac{N-2s}{2}j_1}C(|a_1|^{2s^*-1}|a_2| + |a_1||a_2|^{2s^*-1}) \\ &\leq C(|a_1|^{2s^*-1}|a_2| + |a_1||a_2|^{2s^*-1}), \end{aligned}$$

Therefore we can estimate as follows

$$\begin{aligned} |F(a_1 + a_2) - F(a_1) - F(a_2)| &\leq \\ |F(b_1 + a_2) - F(b_1) - F(a_2)| &+ |F(a_1 + a_2) - F(b_1 + a_2) + F(b_1) - F(a_1)|, \end{aligned}$$

with

$$|F(a_1 + a_2) - F(a_1) - F(b_1 + a_2) + F(b_1)| \leq 2C|a_2| \leq C|a_1|^{2s^*}|a_2|.$$

Case 3: Suppose that $|a_1|, |a_2| \leq 1$. Since

$$\mathbb{R} = \bigcup_{j \in \mathbb{Z}} I_j^- \cup I_j^+,$$

where $I_j^- = [-\gamma^{\frac{N-2s}{2}j}, -\gamma^{\frac{N-2s}{2}(j-1)}]$ and $I_j^+ = [\gamma^{\frac{N-2s}{2}(j-1)}, \gamma^{\frac{N-2s}{2}j}]$ there exists $j_0 \in \mathbb{Z}$ such that

$$\gamma^{\frac{N-2s}{2}j_0}(a_1 + a_2) \in \left[-\gamma^{\frac{N-2s}{2}k}, -\gamma^{\frac{N-2s}{2}(k-1)} \right] \cup \left[\gamma^{\frac{N-2s}{2}(k-1)}, \gamma^{\frac{N-2s}{2}k} \right]$$

Let $b_1 = \gamma^{\frac{N-2s}{2}j_0}a_1$ and $b_2 = \gamma^{\frac{N-2s}{2}j_0}a_2$, with the necessity $|b_1| \geq 1$ or $|b_2| \geq 1$, because $\gamma^{\frac{N-2s}{2}(k-1)} > 2$. Consequently we can use the first or the second case to get that

$$\begin{aligned} \gamma^{Nj_0}|F(a_1 + a_2) - F(a_1) - F(a_2)| &= |F(b_1 + b_2) - F(b_1) - F(b_2)| \\ &\leq \gamma^{Nj_0}C(|a_1|^{2s^*-1}|a_2| + |a_1||a_2|^{2s^*-1}). \end{aligned}$$

The general case follows by a similar argument as above, thus we conclude that (6.4) holds. \square

Remark 6.5. If $f(x, t)$ satisfies (f_6) then

$$\begin{cases} F_0(t) = \lim_{|x| \rightarrow \infty} F(x, t). \\ F_+(t) = \lim_{j \in \mathbb{Z}, j \rightarrow +\infty} \gamma^{-Nj} F\left(\gamma^{-j}x, \gamma^{\frac{N-2s}{2}j}t\right), \\ F_-(t) = \lim_{j \in \mathbb{Z}, j \rightarrow -\infty} \gamma^{-Nj} F\left(\gamma^{-j}x, \gamma^{\frac{N-2s}{2}j}t\right). \end{cases}$$

uniformly in compact sets. Furthermore, $F_+(t)$ and $F_-(t)$ are self-similar.

7. ON THE BEHAVIOR OF WEAK DECOMPOSITION CONVERGENCE UNDER NONLINEARITIES

Concerning the assumptions (f_5) , (f_6) , and (f_9) , we have the following results, which provides a way to link the weak convergence decomposition (as also the latter lines of Theorem 2.1) and the limit over the energy functional I for bounded sequences in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. They are mainly used to prove the existence results stated in Sect. 3. Also, the next result can be seen as a generalization of the well know Brezis-Lieb Lemma [8] (see Corollary 7.3).

Proposition 7.1. *Let $0 < s < \min\{1, N/2\}$ and assume that $f(x, t)$ satisfies (f_1) , (f_2) , (f_5) and (f_6) . Let (u_k) in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ be a bounded sequence and $(w^{(n)})_{n \in \mathbb{N}_*}$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, $n \in \mathbb{N}_*$, provided by Theorem 2.1. Then*

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} F(x, u_k) dx &= \int_{\mathbb{R}^N} F(x, w^{(1)}) dx \\ &+ \sum_{n \in \mathbb{N}_0, n > 1} \int_{\mathbb{R}^N} F_0(w^{(n)}) dx + \sum_{n \in \mathbb{N}_+} \int_{\mathbb{R}^N} F_+(w^{(n)}) dx + \sum_{n \in \mathbb{N}_-} \int_{\mathbb{R}^N} F_-(w^{(n)}) dx. \end{aligned} \quad (7.1)$$

Proof. Let us first introduce the notation

$$d_k^{(n)} u(x) = \gamma^{\frac{N-2s}{2}j_k^{(n)}} u(\gamma^{j_k^{(n)}}(x - y_k^{(n)})), \quad u \in \mathcal{D}^{s,2}(\mathbb{R}^N),$$

where $(y_k^{(n)})_{k \in \mathbb{Z}} \subset \mathbb{Z}^N$, $(j_k^{(n)})_{k \in \mathbb{N}} \subset \mathbb{Z}$. By (f_2) , the functional

$$\Phi(u) = \int_{\mathbb{R}^N} F(x, u) dx, \quad u \in \mathcal{D}^{s,2}(\mathbb{R}^N),$$

is uniformly continuous in bounded sets of $L^{2^*}_s(\mathbb{R}^N)$, which implies (by (2.3) and (2.4)) that

$$\lim_{k \rightarrow \infty} \left[\Phi(u_k) - \Phi\left(\sum_{n \in \mathbb{N}_*} d_k^{(n)} w^{(n)}\right) \right] = 0.$$

To prove (7.1) we observe that the uniform convergence of the series in (2.4) allows us to consider only the case where $\mathbb{N}_* = \{1, \dots, M\}$. Thus,

$$\lim_{k \rightarrow \infty} \left[\sum_{n \in \mathbb{N}_0} \Phi\left(w^{(n)}(\cdot - y_k^{(n)})\right) - \Phi(w^{(1)}) - \sum_{n \in \mathbb{N}_0, n > 1} \Phi_0(w^{(n)}) \right] = 0, \quad (7.2)$$

$$\lim_{k \rightarrow \infty} \left[\sum_{n \in \mathbb{N}_\pm} \Phi(d_k^{(n)} w^{(n)}) - \sum_{n \in \mathbb{N}_\pm} \Phi_\pm(w^{(n)}) \right] = 0, \quad (7.3)$$

follows immediately from the assumption (f_6) , by change of variables and the use of Lebesgue Convergence Theorem. Therefore it is sufficient to prove that

$$\lim_{k \rightarrow \infty} \left[\Phi \left(\sum_{n \in \mathbb{N}_*} d_k^{(n)} w^{(n)} \right) - \sum_{n \in \mathbb{N}_*} \Phi(d_k^{(n)} w^{(n)}) \right] = 0. \quad (7.4)$$

Indeed, by (f_5) we have for all $m \neq n$,

$$\left| \Phi \left(\sum_{n \in \mathbb{N}_*} d_k^{(n)} w^{(n)} \right) - \sum_{n \in \mathbb{N}_*} \Phi(d_k^{(n)} w^{(n)}) \right| \leq \sum_{m \neq n \in \mathbb{N}_*} \int_{\mathbb{R}^N} |d_k^{(n)}|^{2_s^*-1} |d_k^{(m)}| dx.$$

But by a change of variable we can see that

$$\int_{\mathbb{R}^N} |d_k^{(n)}|^{2_s^*-1} |d_k^{(m)}| dx = \int_{\mathbb{R}^N} |w^{(n)}|^{2_s^*-1} g_k(|w^{(m)}|) dx,$$

where

$$g_k(|w^{(m)}|) = \gamma^{\frac{N-2s}{2}(j_k^{(m)} - j_k^{(n)})} w^{(m)} \left(\gamma^{j_k^{(m)} - j_k^{(n)}} (\cdot - \gamma^{j_k^{(n)}} (y_k^{(m)} - y_k^{(n)})) \right) \rightharpoonup 0 \text{ in } \mathcal{D}^{s,2}(\mathbb{R}^N),$$

due to (2.2) and Lemma 5.2. Since

$$\alpha(v) = \int_{\mathbb{R}^N} |w^{(n)}|^{2_s^*-1} v dx$$

is a continuous linear functional in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ we conclude (7.4). \square

Corollary 7.2. *Let (u_k) be a bounded sequence in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ and $(w^{(n)})_{n \in \mathbb{N}_*}$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, $n \in \mathbb{N}_*$, provided by Theorem 2.1. If $F(x, t) = F(t)$ satisfies (f_9) and is locally Lipschitz then, on up to subsequence,*

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} F(u_k) dx = \sum_{n \in \mathbb{N}_*} \int_{\mathbb{R}^N} F(w^{(n)}) dx. \quad (7.5)$$

Proof. In this case $F(t)$ satisfies (f_5) and (6.2), also $F = F_+ = F_- = F_0$. \square

Corollary 7.3. *Let $u_k \rightharpoonup u$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ and $F(t)$ be as in Corollary 7.2 then, up to subsequence,*

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} F(u_k) - F(u - u_k) - F(u) dx = 0.$$

Proof. Since $w^{(1)} = u$, by (2.4) and Corollary 7.2 we have

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} F(u_k - u) dx = \sum_{n \in \mathbb{N}_*, n > 1} \int_{\mathbb{R}^N} F(w^{(n)}) dx. \quad (7.6)$$

Taking the difference between (7.5) and (7.6) we get the desired result. \square

8. THE AUTONOMOUS CASE

The aim of this section is to prove Theorems 3.1, 3.2 and 3.3.

Remark 8.1. By embedding (4.1), we have $\mathcal{S}_l < \infty$. Also \mathcal{S}_l is attained for some l if and only if it is attained for all l . Indeed, this can be checked by considering the rescaling $v(x) = u_1(l^{-1/(N-2s)}x)$ and $u(x) = v_l(l^{1/(N-2s)}x)$, where $\|u_1\| = 1$ and $\|v_l\| = l$ respectively. In particular,

$$l^{\frac{N}{N-2s}} \mathcal{S}_1 = \mathcal{S}_l. \quad (8.1)$$

8.1. Proof of Theorem 3.1.

Proof. Suppose that $F(t)$ is self-similar and satisfies (f'_4) . Let $(u_k) \subset \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a maximizing sequence for (3.4) with $l = 1$, that is, $\|u_k\|^2 = 1$ and $\Phi(u_k) \rightarrow \mathcal{S}_1$. Let $(w^{(n)})_{n \in \mathbb{N}_*}$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, $(y_k^{(n)})_{k \in \mathbb{N}}$ in \mathbb{Z}^N , and $(j_k^{(n)})_{k \in \mathbb{N}}$ in \mathbb{Z} , $n \in \mathbb{N}_*$, be the sequences provided by Theorem 2.1. By the Corollary 7.2,

$$\mathcal{S}_1 = \lim_{k \rightarrow \infty} \Phi(u_k) = \sum_{n \in \mathbb{N}_*} \Phi(w^{(n)}), \quad (8.2)$$

and at the same time by assertion (2.3),

$$\sum_{n \in \mathbb{N}_*} \|w^{(n)}\|^2 \leq \limsup_k \|u_k\|^2 \leq 1. \quad (8.3)$$

The identity (8.2) also implies that there exists $n \in \mathbb{N}_*$ with $w^{(n)} \neq 0$. We may write $v^{(n)}(x) = w^{(n)}(\tau_n x)$ where $\tau_n = \|w^{(n)}\|^{2/(N-2s)}$. Consequently $\|v^{(n)}\|^2 = 1$, $\Phi(v^{(n)}) \leq \mathcal{S}_1$ and

$$\mathcal{S}_1 = \sum_{n \in \mathbb{N}_*} \tau_n^N \Phi(v^{(n)}) \leq \mathcal{S}_1 \sum_{n \in \mathbb{N}_*} \tau_n^N.$$

Moreover,

$$1 \leq \sum_{n \in \mathbb{N}_*} \tau_n^N. \quad (8.4)$$

From (8.3) we have

$$\sum_{n \in \mathbb{N}_*} \tau_n^{N-2s} \leq 1. \quad (8.5)$$

Relations (8.4) and (8.5) can hold simultaneously provided that there is a $n_0 \in \mathbb{N}_*$ such that $\tau_{n_0} = 1$, while $\tau_n = 0$, whenever $n \neq n_0$. Therefore, by (2.4) we obtain

$$u_k - \gamma^{\frac{N-2s}{2} j_k^{(n_0)}} w^{(n_0)} (\gamma^{j_k^{(n_0)}} (\cdot - y_k^{(n_0)})) \rightarrow 0 \text{ in } L^{2s*}(\mathbb{R}^N).$$

Since $F(t)$ is self-similar, the sequence

$$v_k = \gamma^{-\frac{N-2s}{2} j_k^{(n_0)}} u_k (\gamma^{-j_k^{(n_0)}} x + y_k^{(n_0)}),$$

is a maximizing sequence for (3.4) and $v_k \rightarrow w^{(n_0)}$ in $L^{2s*}(\mathbb{R}^N)$. Furthermore, the continuity of Φ in $L^{2s*}(\mathbb{R}^N)$ implies $\Phi(w^{(n_0)}) = \mathcal{S}_1$, and since $\|w^{(n_0)}\|^2 = 1$, $w^{(n_0)}$ is a maximizer.

Consider now the case where $\mathcal{S}_1 > \max\{\mathcal{S}_{1,+}, \mathcal{S}_{1,-}\}$. Let again (u_k) be a maximizing sequence for (3.4) with $l = 1$. Since $f(t)$ verify (f_6) we can apply to Proposition 7.1 to get

$$\mathcal{S}_1 = \lim_{k \rightarrow \infty} \Phi(u_k) = \sum_{n \in \mathbb{N}_0} \Phi(w^{(n)}) + \sum_{n \in \mathbb{N}_{-\infty}} \Phi_{-}(w^{(n)}) + \sum_{n \in \mathbb{N}_{+\infty}} \Phi_{+}(w^{(n)}), \quad (8.6)$$

where $(w^{(n)})$, $(y_k^{(n)})$, $(j_k^{(n)})$, $n \in \mathbb{N}_*$, are given by Theorem 2.1. Considering again $v^{(n)}(x) = w^{(n)}(\tau_n x)$, with $\tau_n = \|w^{(n)}\|^{2/(N-2s)}$, we get

$$1 \leq \sum_{n \in \mathbb{N}_0} \tau_n^N + \frac{\mathcal{S}_{1,-}}{\mathcal{S}_1} \sum_{n \in \mathbb{N}_{-\infty}} \tau_n^N + \frac{\mathcal{S}_{1,+}}{\mathcal{S}_1} \sum_{n \in \mathbb{N}_{+\infty}} \tau_n^N. \quad (8.7)$$

Since $\mathcal{S}_{1,+}/\mathcal{S}_1 < 1$ and $\mathcal{S}_{1,-}/\mathcal{S}_1 < 1$ by assertion (2.3), inequalities (8.3) and (8.7) can hold simultaneously if and only if there is a $n_0 \in \mathbb{N}_0$ such that $\tau_{n_0} = 1$, while $\tau_n = 0$, whenever $n \neq n_0$. Therefore, by assertion (2.4) $u_k - w^{(n_0)}(\cdot - y_k^{(n_0)}) \rightarrow 0$ in $L^{2s*}(\mathbb{R}^N)$ and using a similar argument as in the previous case, we conclude that $w^{(n_0)}$ is a maximizer. \square

Remark 8.2. One always has $\mathcal{S}_1 \geq \max\{\mathcal{S}_{1,+}, \mathcal{S}_{1,-}\}$. Indeed, as discussed above, it suffices to prove this in the case that $l = 1$, so let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ with $\|u\| = 1$ and $v_j := \delta_j u$, where δ_j is given in (5.1), and $j \in \mathbb{Z}$. Then $\|v_j\| = 1$ implies that $\Phi(v_j) \leq \mathcal{S}_1$, and by condition (f₆) we conclude $\Phi(v_j) \rightarrow \Phi_+(u)$ as $j \rightarrow +\infty$. The case for the inequality $\mathcal{S}_1 \geq \mathcal{S}_{1,-}$ follows by using the same argument. Moreover, the inequality $\mathcal{S}_1 > \max\{\mathcal{S}_{1,+}, \mathcal{S}_{1,-}\}$ holds whenever $F \geq F_+$ and $F \geq F_-$ with the strict inequality in a neighborhood of zero. In fact, since F_+ and F_- are self-similar, we may consider w_+ and w_- the maximizers of $\mathcal{S}_{l,+}$ and $\mathcal{S}_{l,-}$, respectively, to obtain, by Theorem 3.2, Proposition 4.4 and Remark 8.1, that $\mathcal{S}_{l,+} < \Phi(w_+) \leq \mathcal{S}_l$ and $\mathcal{S}_{l,-} < \Phi(w_-) \leq \mathcal{S}_l$.

8.2. Characterization of the minimax level. We pass now to the study of the minimax level of the Lagrangian associated with Eq. (\mathcal{P}_s), proving some useful results. This is made by considering the class of paths $\zeta : [0, +\infty) \rightarrow \mathcal{D}^{s,2}(\mathbb{R}^N)$ defined by $\zeta_u(t)(x) = u(x/t)$ for any $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$, because of its homogeneous property with respect to the norm in $\mathcal{D}^{s,2}(\mathbb{R}^N)$.

Lemma 8.3. *Suppose that $F(t)$ satisfies the growing condition (6.2). If $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ is such that $\Phi(u) > 0$, then the path ζ_u belongs to Γ_I . Thus $\Gamma_I \neq \emptyset$ if and only if (f'₄) holds.*

Proof. Let $t_n, t_0 > 0$, $n \in \mathbb{N}$, be such that $t_n \rightarrow t_0$ and $u \in \mathcal{S}_0$. Since

$$\|\zeta_u(t)\|^2 = t^{N-2s}\|u\|^2, \quad \forall t > 0, \quad (8.8)$$

using (5.2) we have

$$\begin{aligned} \|\zeta_u(t_n) - \zeta_u(t_0)\|^2 = \\ t_n^{N-2s}\|u\|^2 - 2t_n^{-s}t_0^{-s} \int_{\mathbb{R}^N} (-\Delta)^{s/2}u(x/t_n)(-\Delta)^{s/2}u(x/t_0) dx + t_0^{N-2s}\|u\|^2. \end{aligned} \quad (8.9)$$

Also, up to a set of Lebesgue measure zero, by identity (4.6) we obtain

$$\begin{aligned} |(-\Delta)^s u(x/t_0)u(x/t_n)| &\leq \|u\|_{L^\infty(\mathbb{R}^N)} |(-\Delta)^s u(x/t_0)|, \\ \lim_{n \rightarrow \infty} (-\Delta)^s u(x/t_0)u(x/t_n) &= (-\Delta)^s u(x/t_0)u(x/t_0), \quad \forall x \in \mathbb{R}^N. \end{aligned}$$

Thus by the Dominated Convergence Theorem the left-hand side of the identity (8.9) goes to zero as $n \rightarrow \infty$. By identity (8.8) we conclude $\zeta_u \in C([0, \infty), \mathcal{D}^{s,2}(\mathbb{R}^N))$. The general case follows by a density argument.

Now suppose that (f'₄) holds. Then there exists $u \in C_0^\infty(\mathbb{R}^N)$ such that $\Phi(u) > 0$ and consequently $\zeta_u \in \Gamma_I$, since

$$I(\zeta_u(t)) = \frac{1}{2}t^{N-2s}\|u\|^2 - t^N\Phi(u) \rightarrow -\infty \text{ as } t \rightarrow \infty.$$

Conversely, assume that $\Gamma_I \neq \emptyset$. If (f'₄) does not hold, then we would have that $I(u) \geq 0$, for all $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$. Hence $\Gamma_I = \emptyset$, which is impossible. \square

Remark 8.4. Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ be such that $\Phi(u) > 0$. Then

$$\max_{t \geq 0} I(\zeta_u(t)) = \frac{1}{2} \left(\frac{\|u\|^2}{2_s^* \Phi(u)} \right)^{\frac{N-2s}{2s}} \|u\|^2 - \left(\frac{\|u\|^2}{2_s^* \Phi(u)} \right)^{N/2s} \Phi(u). \quad (8.10)$$

Lemma 8.5. *Assume that conditions (f'₄) and (6.2) holds. Consider*

$$\tilde{c}(I) := \inf_{\zeta \in \tilde{\Gamma}_I} \sup_{t \geq 0} I(\zeta(t)).$$

where

$$\tilde{\Gamma}_I := \{\zeta \in \Gamma_I : \zeta = \zeta_u \text{ for some } u \in \mathcal{D}^{s,2}(\mathbb{R}^N) \text{ with } \Phi(u) > 0\}$$

Then $c(I) = \tilde{c}(I)$.

Proof. Since $\tilde{\Gamma} \subset \Gamma$ we have $c(I) \leq \tilde{c}(I)$. Suppose the contrary, that $c(I) < \tilde{c}(I)$. Then, there exists $\zeta \in \Gamma_I$ such that $c(I) \leq \sup_{t \geq 0} I(\zeta(t)) < \tilde{c}(I)$. Observe now that, by (4.1) and (f₂), the continuous function

$$h(t) = \frac{1}{2} \|\zeta(t)\|^2 - \frac{2_s^*}{2} \Phi(\zeta(t)), \quad t > 0,$$

changes sign. Hence, there exists $t_0 > 0$ such that $g(t_0) = 0$ and $\zeta(t_0) \neq 0$, which implies that $\|\zeta(t_0)\|^2 = 2_s^* \Phi(\zeta(t_0))$. Now taking $u = \zeta(t_0)$ in (8.10) we get

$$\sup_{t \geq 0} I(\zeta_u(t)) = \frac{1}{2} \|\zeta(t_0)\|^2 - \Phi(\zeta(t_0)) \leq \sup_{t \geq 0} I(\zeta(t)),$$

which leads to a contradiction with the definition of $\tilde{c}(I)$. \square

Remark 8.6. In order to prove our nonlocal counterpart of [42, Proposition 2.4], we have to reduce the class of admissible paths. This is made by noticing that

$$\sup_{t \geq 0} I(\zeta_v(t)) = \sup_{t \geq 0} I(\zeta_{v_\sigma}(t)),$$

for any rescaling $v_\sigma(x) = v(x/\sigma)$, $\sigma > 0$, and taking account the set

$$\tilde{\Gamma}_I^1 := \{ \zeta \in \Gamma_I : \zeta = \zeta_u \text{ for some } u \in \mathcal{D}^{s,2}(\mathbb{R}^N) \text{ with } \Phi(u) > 0 \text{ and } \|u\| \geq 1 \},$$

and the associated minimax level

$$\tilde{c}_1(I) := \inf_{\zeta \in \tilde{\Gamma}_I^1} \sup_{t \geq 0} I(\zeta(t)),$$

to obtain that $\tilde{c}(I) = \tilde{c}_1(I)$.

8.3. Proof of Theorem 3.2.

Proof. (i) Let $v \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a nontrivial critical point of I . By Proposition 4.4 we have $\zeta_v \in \Gamma_I$ and $t = 1$ is a maximum point for the function $t \mapsto I(\zeta_v(t)) = t^{N-2s} \|v\|^2/2 - t^N \Phi(v)$. Hence $c(I) \leq \max_{t \geq 0} I(\zeta_v(t)) = I(v)$.

(ii) Since w is a maximizer for (3.4) we have

$$\int_{\mathbb{R}^N} f(w)v \, dx = \lambda \int_{\mathbb{R}^N} (-\Delta)^{s/2} w (-\Delta)^{s/2} v \, dx, \quad \forall v \in \mathcal{D}^{s,2}(\mathbb{R}^N), \quad (8.11)$$

where λ is a Lagrange multiplier. We claim that $\lambda \neq 0$. Indeed, on the contrary, we get $f(w) = 0$ a.e in \mathbb{R}^N , which leads to a contradiction with $\Phi(w) > 0$. Thus, we can apply Proposition 4.4 to get

$$\lambda \|w\|^2 = 2_s^* \int_{\mathbb{R}^N} F(w) \, dx,$$

which together with relation (8.1) implies $\lambda l_0 = 2_s^* S_1 l_0^{N/(N-2s)}$, and the explicit value of l_0 gives $\lambda = 1$. In particular,

$$I(w) = \left(\frac{1}{2} - \frac{1}{2_s^*} \right) \|w\|^2 > 0.$$

Let us prove now the last statement of (ii). By the part (i), it is sufficient to prove that $I(w) \leq c(I)$. Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ with $\Phi(u) > 0$, and denote $\tilde{u}(x) = u(\alpha x)$ where $\alpha = \|u\|^{2/(N-2s)}$. Then $\|\tilde{u}\| = 1$ and consequently

$$\Phi(\zeta_u(t)) = \Phi(\zeta_{\tilde{u}}(t\alpha)) \leq \|\zeta_u(t)\|^{\frac{2N}{N-2s}} S_1, \quad \forall t \geq 0,$$

from which we can deduce, by Lemma 8.5 and Remark 8.6, that

$$\begin{aligned} c(I) &= \inf_{\substack{\Phi(u) > 0, \\ \|u\| \geq 1}} \sup_{t \geq 0} \left[\frac{1}{2} \|\zeta_u(t)\|^2 - \Phi(\zeta_u(t)) \right] \\ &\geq \inf_{\substack{\Phi(u) > 0, \\ \|u\| \geq 1}} \sup_{t \geq 0} \left[\frac{1}{2} \|\zeta_u(t)\|^2 - \|\zeta_u(t)\|^{\frac{2N}{N-2s}} \mathcal{S}_1 \right]. \end{aligned}$$

Moreover, we have

$$\begin{aligned} &\sup_{t \geq 0} \left[\frac{1}{2} \|\zeta_u(t)\|^2 - \|\zeta_u(t)\|^{\frac{2N}{N-2s}} \mathcal{S}_1 \right] \\ &= \left[\frac{1}{2} (2_s^* \mathcal{S}_1)^{-\frac{N-2s}{2s}} - \mathcal{S}_1 (2_s^* \mathcal{S}_1)^{-\frac{N}{2s}} \right] \|u\|^{2_s^*(1-s)}, \quad \forall u \in D^{s,2}(\mathbb{R}^N) \text{ with } \Phi(u) > 0. \end{aligned}$$

Consequently,

$$\inf_{\substack{\Phi(u) > 0, \\ \|u\| \geq 1}} \sup_{t \geq 0} \left[\frac{1}{2} \|\zeta_u(t)\|^2 - \|\zeta_u(t)\|^{\frac{2N}{N-2s}} \mathcal{S}_1 \right] = \frac{1}{2} (2_s^* \mathcal{S}_1)^{-\frac{N-2s}{2s}} - \mathcal{S}_1 (2_s^* \mathcal{S}_1)^{-\frac{N}{2s}}.$$

On the other hand, by the explicit value of l_0 and relation (8.1) we have that

$$I(w) = \frac{1}{2} (2_s^* \mathcal{S}_1)^{-\frac{N-2s}{2s}} - \mathcal{S}_1 (2_s^* \mathcal{S}_1)^{-1\frac{N}{2s}}.$$

Thus $c(I) = I(w)$ and by the proof of the statement (i), the path $\zeta_w \in \Gamma_I$ is minimal. \square

8.4. Proof of Theorem 3.3.

Proof. We start by noting that the embedding (4.1) together with condition (f₂) implies $\mathcal{I} > 0$. Let (u_k) be a minimizing sequence, that is, $\Phi(u_k) = 1$ and $\|u_k\|^2 \rightarrow \mathcal{I}$. Since this sequence is bounded, we may apply Theorem 2.1 to obtain the weak profile described in (2.1)–(2.4). By the Corollary 7.2, we have

$$1 = \sum_{n \in \mathbb{N}_*} \int_{\mathbb{R}^N} F(w^{(n)}) \, dx,$$

which implies that there exists $n \in \mathbb{N}_*$ with $0 < \Phi(w^{(n)}) \leq 1$. If $\Phi(w^{(n)}) = 1$, considering d_k as the element of $D_{\mathbb{Z}^N, \mathbb{R}}$ given by assertion (2.1), we have by the weak lower semi-continuity of the norm that

$$\mathcal{I} \leq \|w^{(n)}\|^2 \leq \liminf_{k \rightarrow \infty} \|d_k^* u_k\|^2 = \mathcal{I} \quad \text{and} \quad \|d_k^* u_k\|^2 = \|u_k\|^2 \rightarrow \|w^{(n)}\|^2,$$

which proves the first part of Theorem 3.3. Hence, let us assume that $\Phi(w^{(n)}) < 1$. Set $v_k = d_k^* u_k - w$, where $w = w^{(n)}$. By Corollary 7.3 we have

$$\lim_{k \rightarrow \infty} \left[1 - \int_{\mathbb{R}^N} F(v_k) \, dx \right] = \int_{\mathbb{R}^N} F(w) \, dx \quad (8.12)$$

Denote $\delta = \Phi(w)$ and set $\hat{w}(x) = w(\delta^{1/N} x)$. Thus $\Phi(\hat{w}) = 1$ and consequently

$$\|w\|^2 = \delta^{\frac{N-2s}{N}} \|\hat{w}\|^2 \geq \delta^{\frac{N-2s}{N}} \mathcal{I}. \quad (8.13)$$

Now consider

$$\hat{v}_k(x) = v_k(|1 - \delta|^{1/N} \beta_k^{1/N} x), \quad \text{where } \beta_k = \Phi(b_k) \text{ and } b_k(x) = v_k(|1 - \delta|^{1/N} x).$$

Since $\beta_k = |1 - \delta|^{-1} \Phi(v_k)$, by convergence (8.12) we have $\beta_k \rightarrow 1$, and we conclude $\Phi(\hat{v}_k) = 1$ for large k . This leads to

$$\|v_k\|^2 = |1 - \delta|^{\frac{N-2s}{N}} \beta_k^{\frac{N-2s}{N}} \|\hat{v}_k\|^2 \geq |1 - \delta|^{\frac{N-2s}{N}} \beta_k^{\frac{N-2s}{N}} \mathcal{I}, \quad (8.14)$$

for large k . In the other hand, since $\|u_k\|^2 = \|d_k^* u_k\|^2$, by relations (8.13) and (8.14) we may infer

$$\begin{aligned} \|u_k\|^2 &= \|v_k\|^2 + 2(v_k, w) + \|w\|^2 \\ &\geq \left(\delta^{\frac{N-2s}{N}} + |1 - \delta|^{\frac{N-2s}{N}} \beta_k^{\frac{N-2s}{N}} \right) \mathcal{I}, \end{aligned}$$

and passing the limit we finally conclude

$$1 \geq \delta^{1-\frac{2s}{N}} + |1 - \delta|^{1-\frac{2s}{N}},$$

which leads to a contradiction since $0 < \delta < 1$. Thus w is the minimizer in (3.5) and consequently we have

$$\int_{\mathbb{R}^N} (-\Delta)^{s/2} w (-\Delta)^{s/2} v \, dx = \lambda \int_{\mathbb{R}^N} f(w) v \, dx, \quad \forall v \in \mathcal{D}^{s,2}(\mathbb{R}^N),$$

where $\lambda \in \mathbb{R}$ is a Lagrange multiplier. Taking $v = w$ in the above identity we have $\lambda \neq 0$, which allows us to apply Proposition 4.4 to get $\lambda = \mathcal{I}/2_s^*$, which by an easy computation using identities (5.2) leads us to conclude that $u(x) = w(x/\beta)$ is a non-trivial weak solution of Eq. (\mathcal{P}_s), where $\beta = \lambda^{1/2s} = (\mathcal{I}/2_s^*)^{1/2s}$.

Let us prove now that $u(x) = w(x/\beta)$ is a ground state solution of Eq. (\mathcal{P}_s). We start by applying Proposition 4.4 again to obtain

$$I(u) = \left(\frac{1}{2} - \frac{1}{2_s^*} \right) \|u\|^2 = \frac{s}{N} (2_s^*)^{-\frac{N-2s}{2s}} \|w\|^{N/s}. \quad (8.15)$$

Now let $v \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ be any non-trivial weak solution of Eq. (\mathcal{P}_s). For any $\sigma > 0$ denote $v_\sigma(x) = v(x/\sigma)$. Choose σ such that $\Phi(v_\sigma) = 1$, that is, $\sigma = (\Phi(v))^{-1/N}$. Replacing this value of $\Phi(v)$ in the identity $\|v\|^2 = 2_s^* \Phi(v)$, we get $\sigma = (2_s^*)^{1/N} \|v\|^{-2/N}$. Consequently, we obtain

$$\|v_\sigma\|^2 = (2_s^*)^{\frac{N-2s}{N}} (\|v\|^2)^{2s/N},$$

which implies

$$I(v) = \frac{s}{N} \|v\|^2 = \frac{s}{N} (2_s^*)^{-\frac{N-2s}{2s}} \|v_\sigma\|^{N/s}. \quad (8.16)$$

Comparing identities (8.15) and (8.16), we conclude that $I(u) \leq I(v)$, i.e, u is a ground state solution for Eq. (\mathcal{P}_s). \square

9. THE NON-AUTONOMOUS CASE

For the sake of discussion, we are going to compare the minimax level of the asymptotic functional I_κ , with the minimax of the Lagrangian associated with Eq. (\mathcal{P}_s), for $\kappa = 0, +, -$.

Proposition 9.1. *Suppose that (f₁)–(f₇) holds. If F_0 is self-similar or $(F_0)_\kappa(t) \leq F_\kappa(t)$, for all t , $\kappa = +, -$, then $c(I) \leq c(I_\kappa)$, for $\kappa = 0, +, -$. Moreover, under these assumptions, (f₈^l) implies (f₈).*

Proof. Let be \mathcal{S}_l^κ , the associated constrained maximum similar to (3.4) relative to the primitive F_κ , precisely,

$$\mathcal{S}_l^\kappa = \sup_{\|u\|^2=l} \int_{\mathbb{R}^N} F_\kappa(u) \, dx \quad \text{for } \kappa = 0, +, -.$$

For each $\kappa = +, -$, the primitive of the nonlinearity F_κ is auto-similar, thus using Theorems 3.1 and 3.2, we conclude that there exists w_κ maximizer of $\mathcal{S}_{l_0}^\kappa$ such that

$$c(I_\kappa) = I_\kappa(w_\kappa) = \max_{t \geq 0} I_\kappa(\zeta_{w_\kappa}(t)) > 0.$$

For each $\kappa = +, -$, let us consider the sequence

$$w_n^\kappa(x) := \gamma^{\frac{N-2s}{2}j_n^\kappa} w_\kappa(\gamma^{j_n^\kappa} x),$$

where the sequence $(j_n^\kappa) \subset \mathbb{Z}$ is chosen in such a way that $j_n^+ \rightarrow +\infty$ and $j_n^- \rightarrow -\infty$. Since for each $\kappa = +, -$,

$$|I(\zeta_{w_n^\kappa}(t)) - I_\kappa(\zeta_{w_\kappa}(t))| \leq t^N \int_{\mathbb{R}^N} \left| \gamma^{-Nj_n^\kappa} F\left(\gamma^{-j_n^\kappa} tx, \gamma^{\frac{N-2s}{2}j_n^\kappa} w_\kappa(x)\right) - F_\kappa(w_\kappa(x)) \right| dx, \quad (9.1)$$

the uniformity assumption on the limits in (f6), guarantees (by a density argument) that

$$\lim_{n \rightarrow \infty} I(\zeta_{w_n^\kappa}(t)) = I_\kappa(\zeta_{w_\kappa}(t)), \quad \text{uniformly in compact sets of } \mathbb{R}. \quad (9.2)$$

We also have that the path $\zeta_{w_n^\kappa}$, $\kappa = +, -$, belongs to Γ_I , for n large enough. In fact, by the uniformly convergence in x of (f6) and Proposition 4.4, there exists $n_0 > 0$ such that

$$\int_{\mathbb{R}^N} \gamma^{-Nj_n^\kappa} F\left(\gamma^{-j_n^\kappa} tx, \gamma^{\frac{N-2s}{2}j_n^\kappa} w_\kappa(x)\right) dx > \frac{1}{2} \int_{\mathbb{R}^N} F_\kappa(w_\kappa) dx, \quad \text{for all } n > n_0 \text{ and } t > 0.$$

Thus, for each m there exist $t_m > 0$ such that

$$I(\zeta_{w_n^\kappa}(t_m)) = \max_{t \geq 0} I(\zeta_{w_n^\kappa}(t)) > 0.$$

We claim that the sequence (t_m) is bounded. On the contrary, up to subsequence, we get the following contradiction

$$0 < I(\zeta_{w_n^\kappa}(t_m)) = \frac{1}{2} t_m^{N-2s} [w]_s^2 - t_m^N \int_{\mathbb{R}^N} \gamma^{-Nj_n^\kappa} F\left(\gamma^{-j_n^\kappa} t_m x, \gamma^{\frac{N-2s}{2}j_n^\kappa} w_\kappa(x)\right) dx \rightarrow -\infty, \quad \text{as } m \rightarrow \infty.$$

Therefore, up to subsequence, $t_m \rightarrow t_0$, and we have

$$\lim_{m \rightarrow \infty} \max_{t \geq 0} I(\zeta_{w_n^\kappa}(t)) = I_\kappa(\zeta_{w_\kappa}(t_0)),$$

because of (9.2). Thus we may conclude

$$c(I) \leq \lim_{n \rightarrow \infty} \max_{t \geq 0} I(\zeta_{w_n^\kappa}(t)) \leq \max_{t \geq 0} I_\kappa(\zeta_{w_\kappa}(t)) = c(I_\kappa).$$

If there exists maximizer w_0 for $\mathcal{S}_{l_0}^0$, then an similar argument as above leads to $c(I) \leq c(I_0)$. In fact, for each n , define the path

$$\lambda_n(t) = w_0\left(\frac{\cdot - y_n}{t}\right), \quad t \geq 0,$$

where (y_n) is taken in a such way that $|y_n| \rightarrow \infty$. As before, we consider the estimate

$$|I(\lambda_n(t)) - I_0(w_0(\cdot/t))| \leq t^N \int_{\mathbb{R}^N} |F(tx + y_n, w_0) - F_0(w_0)| dx,$$

to obtain that

$$\lim_{n \rightarrow \infty} I(\lambda_n(t)) = I_0(w_0(\cdot/t)), \quad \text{uniformly in compact sets of } \mathbb{R}.$$

We also have that the path λ_n belongs to Γ_I , for n large enough. Indeed, assuming the contrary, we would obtain n_0 and a sequence $l_n \rightarrow \infty$ such that $I(\lambda_{n_0}(l_n)) > 0$, for all n . On the other hand, we have that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} F(l_n x + y_{n_0}, w_0) dx = \int_{\mathbb{R}^N} F_0(w_0) dx,$$

which, by taking n large enough, leads to the contradiction $I(\lambda_{n_0}(l_n)) < 0$. Let $t_n > 0$ such that

$$I(\lambda_n(t_n)) = \max_{t \geq 0} I(\lambda_n(t)) > 0.$$

Once again we get that the sequence (t_n) is bounded. On the contrary, up to subsequence, we may assume without loss of generality that $t_n \geq |y_n|^2$ and get the following contradiction

$$0 < I(\lambda_n(t_n)) = \frac{1}{2} t_n^{N-2s} [w_0]_s^2 - t_n^N \int_{\mathbb{R}^N} F(t_n x + y_n, w_0) dx \rightarrow -\infty, \text{ as } n \rightarrow \infty.$$

Thus, up to subsequence, $t_n \rightarrow t_0$ and we obtain that

$$\lim_{n \rightarrow \infty} \max_{t \geq 0} I(\lambda_n(t)) = I_0(w_0(\cdot/t_0)).$$

As a consequence we conclude that

$$c(I) \leq \lim_{n \rightarrow \infty} \max_{t \geq 0} I(\lambda_n(t_n)) \leq \max_{t \geq 0} I_0(w_0(\cdot/t)) = c(I_0),$$

where we used Proposition 4.4 to induce that $t = 1$ is the unique critical point of $I_0(w_0(\cdot/t))$. Thus, let us assume that $\mathcal{S}_{l_0}^0$ is not attained. By the Remarks 8.1 and 8.2; and Theorem 3.1, if $\mathcal{S}_{l_0}^0$ is not attained then $\mathcal{S}_{l_0}^0 = \mathcal{S}_{l_0}^+$ or $\mathcal{S}_{l_0}^0 = \mathcal{S}_{l_0}^-$. Thus, using the definition of $\mathcal{S}_{l_0}^0$ we get

$$c(I_\kappa) \leq I_0(u), \quad \kappa = +, -, \quad \forall u \in \mathcal{D}^{s,2}(\mathbb{R}^N) \text{ with } \|u\|^2 = l_0.$$

Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$, $u \neq 0$, and denote $\alpha = \|u\|^2$, then considering the rescaling $u_{l_0}(x) = u(t_0 x)$, where $t_0 = (\alpha/l_0)^{-1/(N-2s)}$, we have $\|u_{l_0}\|^2 = l_0$ and consequently

$$\begin{aligned} c(I_\kappa) &\leq I(u_{l_0}) = \frac{1}{2} t_0^{N-2s} \|u\|^2 - t_0^N \Phi_0(u) \\ &\leq \max_{t \geq 0} I_0(\zeta_u(t)), \text{ for } \kappa = +, -. \end{aligned}$$

By Lemma 8.5 we conclude $c(I_\kappa) \leq c(I_0)$, $\kappa = +, -$.

Now suppose that (f'_8) holds. As seen above, ζ_{w_κ} belongs to Γ_I , thus

$$c(I) \leq \max_{t \geq 0} I(\zeta_{w_\kappa}(t)) < \max_{t \geq 0} I_\kappa(\zeta_{w_\kappa}(t)) = c(I_\kappa), \quad \kappa = +, -.$$

We claim that $\mathcal{S}_{l_0}^0$ is attained, from which we conclude the desired inequality in (f_8) . Assume the contrary, by arguing as before, we have $\mathcal{S}_{l_0}^0 = \mathcal{S}_{l_0}^+$ or $\mathcal{S}_{l_0}^0 = \mathcal{S}_{l_0}^-$. Taking $|x| \rightarrow \infty$ in (f'_8) we get that $F_0(t) \geq F_\kappa(t)$, $\kappa = +, -$, for all $t \in \mathbb{R}$. Consequently, in any case,

$$\begin{aligned} \int_{\mathbb{R}^N} F_0(w_\kappa) dx &\leq \sup_{\|u\|^2 = l_0} \int_{\mathbb{R}^N} F_0(u) dx \\ &= \int_{\mathbb{R}^N} F_\kappa(w_\kappa) dx \leq \int_{\mathbb{R}^N} F_0(w_\kappa) dx, \quad \kappa = +, -, \end{aligned} \tag{9.3}$$

a contradiction, because relation (9.3) implies that $\mathcal{S}_{l_0}^0$ is attained. \square

Summarizing all the discussion until now we can finally prove Theorem 3.4.

9.1. Proof of Theorem 3.4. In order to treat the case without compactness condition (f₈), that is not considered in the local counterpart [42], where the case $c(I_\kappa) = c(I)$, $\kappa = 0, +, -$, may occur, we need the following result, which states that the existence of a critical point of I is guaranteed whenever the minimax level (3.2) is attained.

Theorem G. [25, Theorem 2.3] *Let E be a real Banach space. Suppose that $I \in C^1(E)$ satisfies*

- (i) $I(0) = 0$;
- (ii) *There exists $r, b > 0$ such that $I(u) \geq b$, whenever $\|u\| = r$;*
- (iii) *There is $e \in E$ with $\|e\| > r$ and $I(e) < 0$;*

Let

$$c(I) = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \|\gamma(1)\| > r, I(\gamma(1)) < 0\}.$$

If there exists $\gamma_0 \in \Gamma$ such that

$$c = \max_{t \in [0,1]} I(\gamma_0(t)),$$

then I possess a nontrivial critical point $u \in \gamma_0([0,1])$ such that $I(u) = c$.

Remark 9.2. We define

$$c_1(I) = \inf_{\gamma \in \Gamma_I} \sup_{t \in [0,1]} I(\gamma(t)),$$

where

$$\Gamma_I^1 = \{\gamma \in C([0,1], \mathcal{D}^{s,2}(\mathbb{R}^N)) : \gamma(0) = 0, \|\gamma(1)\| > r, I(\gamma(1)) < 0\},$$

as the usual minimax level. We have that $c_1(I) = c(I)$.

Proof of Theorem 3.4 completed. For the reader convenience, we divide the proof in several steps.

(i) Let us assume first that condition (f₈) holds true. We start observing that the assumptions (f₃) and (f₄) implies that the functional I has the mountain pass geometry. In particular, $\Gamma_I \neq \emptyset$ and $0 < c(I) < \infty$. In fact, set $v = \varphi_R(x) := v_R(|x - x_0|)$, where v_R as defined as in Remark 3.6. Then $\varphi_R \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ and we have

$$\begin{aligned} \int_{\mathbb{R}^N} F(x, v) dx &= \int_{B_R(x_0)} F(x, t_0) dx + \int_{B_{R+1}(x_0) \setminus B_R(x_0)} F(x, v) dx \\ &\geq |B_R| \inf_{B_R(x_0)} F(x, t_0) + |B_{R+1} \setminus B_R| \inf_{(x,t) \in (B_{R+1}(x_0) \setminus B_R(x_0)) \times [0,t_0]} F(x, t) > 0 \end{aligned}$$

Since (f₃) is equivalent to $d/dt(F(x, t)t^{-\mu}) \geq 0$, $t > 0$, we have for $t > 1$ that

$$\int_{\mathbb{R}^N} F(x, tv) dx \geq t^\mu \int_{\mathbb{R}^N} F(x, v) dx.$$

Hence

$$I(tv) = \frac{t^2}{2} \|v\|^2 - \int_{\mathbb{R}^N} F(x, tv) dx \leq \frac{t^2}{2} \|v\|^2 - t^\mu \int_{\mathbb{R}^N} F(x, v) dx \rightarrow -\infty, \text{ as } t \rightarrow \infty.$$

In the other hand, by the growth condition (f₂) and the embedding (4.1),

$$I(u) \geq \|u\|^2 \left(\frac{1}{2} - C\|u\|^{2_s^*-2} \right), \quad u \in \mathcal{D}^{s,2}(\mathbb{R}^N),$$

for some constant $C > 0$. Thus, choosing $\|u\|$ sufficiently small, we have $I(u) > 0$. The same can be concluded for the functionals I_κ , since F_κ satisfies (f₃) and (f₄).

Let (u_k) in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ be such that $I(u_k) \rightarrow c(I)$ and $I'(u_k) \rightarrow 0$, which the existence can be guaranteed by the Mountain Pass Theorem (see [1]).

(ii) By assumption (f₃), this sequence is bounded in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, since for large k , we have

$$\begin{aligned} c(I) + 1 + \|u_k\| &\geq I(u_k) - \frac{1}{\mu} I'(u_k) \cdot u_k \\ &= \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u_k\|^2 - \int_{\mathbb{R}^N} F(x, u_k) - \frac{1}{\mu} f(x, u_k) u_k \, dx \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu} \right) \|u_k\|^2. \end{aligned}$$

Let $(w^{(n)})$, $(y_k^{(n)})$ and $(j_k^{(n)})$ be the sequences provided by Theorem 2.1. If $w^{(n)} = 0$ for all $n \geq 2$, then by assertion (2.4) and (2.1),

$$u_k \rightarrow w^{(1)} \text{ in } L^{2^*}(\mathbb{R}^N) \text{ and } u_k \rightharpoonup w^{(1)} \text{ in } \mathcal{D}^{s,2}(\mathbb{R}^N).$$

Therefore we conclude that $w^{(1)}$ is a critical point of I such that, up to subsequence, $u_k \rightarrow w^{(1)}$ in $\mathcal{D}^{s,2}(\mathbb{R}^N)$.

(iii) Let us argue by contradiction and assume that there exists $n_0 \geq 2$, such that $w^{(n_0)} \neq 0$. By the estimate (2.3) and Proposition 7.1 we have, up to subsequence, that

$$\begin{aligned} c(I) &= \lim_{k \rightarrow \infty} \left[\frac{1}{2} \|u_k\|^2 - \int_{\mathbb{R}^N} F(x, u_k) \, dx \right] \\ &\geq I(w^{(1)}) + \sum_{n \in \mathbb{N}_0, n > 1} I_0(w^{(n)}) + \sum_{n \in \mathbb{N}_+} I_+(w^{(n)}) + \sum_{n \in \mathbb{N}_-} I_-(w^{(n)}). \end{aligned} \quad (9.4)$$

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $n \geq 1$. Since

$$\left| \gamma^{-\frac{N+2s}{2} j_k^{(n)}} f \left(\gamma^{-j_k^{(n)}} x + y_k^{(n)}, \gamma^{\frac{N-2s}{2} j_k^{(n)}} t \right) \right| \leq C |t|^{2^*-1}, \quad \forall x \in \mathbb{R}^N \text{ and } t \in \mathbb{R},$$

by the embedding (4.5), we can take the limit

$$\begin{aligned} &I'(u_k) \cdot \left(\gamma^{\frac{N-2s}{2} j_k^{(n)}} \varphi(\gamma^{j_k^{(n)}} (\cdot - y_k^{(n)})) \right) \\ &= \left(\gamma^{-\frac{N-2s}{2} j_k^{(n)}} u_k(\gamma^{-j_k^{(n)}} \cdot + y_k^{(n)}), \varphi \right) - \int_{\mathbb{R}^N} \gamma^{-\frac{N+2s}{2} j_k^{(n)}} f \left(\gamma^{-j_k^{(n)}} x + y_k^{(n)}, \gamma^{\frac{N-2s}{2} j_k^{(n)}} v_k^{(n)} \right) \varphi \, dx, \end{aligned}$$

where

$$v_k^{(n)}(x) := \gamma^{-\frac{N-2s}{2} j_k^{(n)}} u_k(\gamma^{-j_k^{(n)}} x + y_k^{(n)}),$$

to conclude that $w^{(1)}$ is a critical point of I and $w^{(n)}$ is a critical point of I_0, I_+ or I_- , provided that $n \in \mathbb{N}_0, \mathbb{N}_+$ or \mathbb{N}_- , respectively. Consequently, using assumption (f₃)

$$I_\kappa(w^{(n)}) = \frac{1}{2} \int_{\mathbb{R}^N} f_\kappa(w^{(n)}) w^{(n)} \, dx - \int_{\mathbb{R}^N} F_\kappa(w^{(n)}) \, dx \geq 0, \quad \forall n \geq 2,$$

and $I(w^{(1)}) \geq 0$. On the other hand, the assumption $c(I) < c(I_\kappa)$ and the estimate (9.4) implies $I_\kappa(w^{(n_0)}) < c(I_\kappa)$, which leads to a contradiction with Theorem 3.2.

(iv) Suppose now that relation (3.3) holds instead of (f₈). Condition (3.3) implies that the path $\zeta_{w^{(n_0)}}$ belongs to Γ_I and $c(I) \leq I_\kappa(w^{(n_0)})$, where κ is the corresponding index for which n_0 belongs. In view of the above discussion and estimate (9.4), we conclude that

$$u_k \rightarrow w^{(1)} \text{ in a subsequence} \quad \text{or} \quad c(I) = \max_{t \geq 0} I(w^{(n_0)}(\cdot/t)).$$

If the minimax level $c(I)$ is attained then we can apply Theorem G to obtain the existence of critical point $u \in \zeta_w(n_0)([0, \infty))$ such that $I(u) = c(I)$. \square

REFERENCES

- [1] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Functional Analysis* **14** (1973) 349–381. [8](#), [35](#)
- [2] D. Applebaum, *Lévy processes and stochastic calculus* (Cambridge University Press, Cambridge, 2009). [2](#)
- [3] D. Applebaum, Lévy processes—from probability to finance and quantum groups, *Notices Amer. Math. Soc.* **51** (2004) 1336–1347. [2](#)
- [4] B. Barrios, E. Colorado, A. de Pablo and U. Sánchez, On some critical problems for the fractional Laplacian operator, *J. Differential Equations* **252** (2012) 6133–6162. [13](#)
- [5] H. Berestycki and P.-L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, *Arch. Rational Mech. Anal.* **82** (1983) 313–345. [2](#), [7](#), [8](#), [10](#)
- [6] H. Berestycki and P.-L. Lions, Nonlinear scalar field equations. II. Existence of infinitely many solutions, *Arch. Rational Mech. Anal.* **82** (1983) 347–375. [2](#)
- [7] H. Brézis and T. Kato, Remarks on the Schrödinger operator with singular complex potentials, *J. Math. Pures Appl. (9)* **58** (1979) 137–151. [13](#)
- [8] H. Brézis and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.* **88** (1983) 486–490. [25](#)
- [9] K. Brezis, How to recognize constant functions. A connection with Sobolev spaces, *Uspekhi Mat. Nauk* **57** (2002) 59–74. [10](#)
- [10] L. Caffarelli, Non-local diffusions, drifts and games, *Nonlinear partial differential equations, Abel Symp.*, vol. 7, Springer, Heidelberg (2012) 37–52. [2](#)
- [11] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations* **32** (2007) 1245–1260. [3](#), [10](#), [15](#)
- [12] X. Chang and Z.-Q. Wang, Ground state of scalar field equations involving a fractional Laplacian with general nonlinearity, *Nonlinearity* **26** (2013) 479–494. [2](#), [3](#), [10](#)
- [13] M. Cwikel and K. Tintarev, On interpolation of cocompact imbeddings, *Rev. Mat. Complut.* **26** (2013) 33–55. [4](#)
- [14] J. Dávila, M. del Pino, S. Dipierro and E. Valdinoci, Concentration phenomena for the nonlocal Schrödinger equation with Dirichlet datum, *Anal. PDE* **8** (2015) 1165–1235. [2](#)
- [15] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces, *Bull. Sci. Math.* **136** (2012) 521–573. [1](#), [10](#)
- [16] S. Dipierro, G. Palatucci and E. Valdinoci, Existence and symmetry results for a Schrödinger type problem involving the fractional Laplacian, *Matematiche (Catania)* **68** (2013) 201–216. [8](#)
- [17] E. B. Fabes, C. E. Kenig and R. P. Serapioni, The local regularity of solutions of degenerate elliptic equations, *Comm. Partial Differential Equations* **7** (1982) 77–116. [11](#), [13](#)
- [18] M. M. Fall and V. Felli, Unique continuation property and local asymptotics of solutions to fractional elliptic equations, *Comm. Partial Differential Equations* **39** (2014) 354–397. [3](#), [11](#), [13](#), [17](#)
- [19] P. Felmer, A. Quaas and J. Tan, Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian, *Proc. Roy. Soc. Edinburgh Sect. A* **142** (2012) 1237–1262. [2](#)
- [20] R. L. Frank and E. Lenzmann, Uniqueness of non-linear ground states for fractional Laplacians in \mathbb{R} , *Acta Math.* **210** (2013) 261–318. [3](#)
- [21] P. Gérard, Description du défaut de compacité de l’injection de Sobolev, *ESAIM Control Optim. Calc. Var.* **3** (1998) 213–233 (electronic). [4](#), [5](#)
- [22] S. Jaffard, Analysis of the lack of compactness in the critical Sobolev embeddings, *J. Funct. Anal.* **161** (1999) 384–396. [4](#), [5](#)
- [23] T. Jin, Y. Li and J. Xiong, On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions, *J. Eur. Math. Soc. (JEMS)* **16** (2014) 1111–1171. [3](#), [11](#), [13](#)
- [24] N. S. Landkof, *Foundations of modern potential theory*, Springer-Verlag, New York-Heidelberg (1972), translated from the Russian by A. P. Doohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band 180. [1](#)
- [25] H. F. Lins and E. A. B. Silva, Quasilinear asymptotically periodic elliptic equations with critical growth, *Nonlinear Anal.* **71** (2009) 2890–2905. [8](#), [34](#)

- [26] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. I, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1** (1984) 109–145. [8](#)
- [27] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. II, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **1** (1984) 223–283. [8](#)
- [28] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The limit case. I, *Rev. Mat. Iberoamericana* **1** (1985) 145–201. [8](#)
- [29] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The limit case. II, *Rev. Mat. Iberoamericana* **1** (1985) 45–121. [8](#)
- [30] A. Nekvinda, Characterization of traces of the weighted Sobolev space $W^{1,p}(\Omega, d_M^\varepsilon)$ on M , *Czechoslovak Math. J.* **43(118)** (1993) 695–711. [11](#)
- [31] G. Palatucci and A. Pisante, Improved Sobolev embeddings, profile decomposition, and concentration-compactness for fractional Sobolev spaces, *Calc. Var. Partial Differential Equations* **50** (2014) 799–829. [4](#), [5](#), [19](#), [20](#)
- [32] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations* (Conference Board of the Mathematical Sciences, Washington, 1986). [6](#)
- [33] P. H. Rabinowitz, On a class of nonlinear Schrödinger equations, *Z. Angew. Math. Phys.* **43** (1992) 270–291. [8](#)
- [34] X. Ros-Oton and J. Serra, Fractional Laplacian: Pohozaev identity and nonexistence results, *C. R. Math. Acad. Sci. Paris* **350** (2012) 505–508. [11](#)
- [35] X. Ros-Oton and J. Serra, The Pohozaev identity for the fractional Laplacian, *Arch. Ration. Mech. Anal.* **213** (2014) 587–628. [3](#)
- [36] R. Servadei and E. Valdinoci, Variational methods for non-local operators of elliptic type, *Discrete Contin. Dyn. Syst.* **33** (2013) 2105–2137. [2](#)
- [37] R. Servadei and E. Valdinoci, The Brezis-Nirenberg result for the fractional Laplacian, *Trans. Amer. Math. Soc.* **367** (2015) 67–102. [2](#)
- [38] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, *Comm. Pure Appl. Math.* **60** (2007) 67–112. [1](#), [13](#)
- [39] S. Solimini, A note on compactness-type properties with respect to Lorentz norms of bounded subsets of a Sobolev space, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **12** (1995) 319–337. [4](#)
- [40] M. Struwe, A global compactness result for elliptic boundary value problems involving limiting nonlinearities, *Math. Z.* **187** (1984) 511–517. [4](#)
- [41] K. Tintarev, Positive solutions of elliptic equations with a critical oscillatory nonlinearity, *Discrete Contin. Dyn. Syst.* (2007) 974–981. [3](#), [23](#)
- [42] K. Tintarev, Concentration compactness at the mountain pass level in semilinear elliptic problems, *NoDEA Nonlinear Differential Equations Appl.* **15** (2008) 581–598. [2](#), [3](#), [4](#), [23](#), [29](#), [34](#)
- [43] K. Tintarev and K.-H. Fieseler, *Concentration compactness* (Imperial College Press, London, 2007). [3](#), [4](#), [7](#), [17](#), [18](#), [20](#), [23](#)
- [44] B. O. Turesson, *Nonlinear potential theory and weighted Sobolev spaces*, *Lecture Notes in Mathematics*, vol. 1736, Springer-Verlag, Berlin (2000). [11](#)
- [45] A. Xia and J. Yang, Regularity of nonlinear equations for fractional Laplacian, *Proc. Amer. Math. Soc.* **141** (2013) 2665–2672. [13](#)
- [46] J. Zhang, J. M. do Ó and M. Squassina, Fractional Schrödinger–Poisson Systems with a General Subcritical or Critical Nonlinearity, *Adv. Nonlinear Stud.* **16** (2016) 15–30. [2](#)

(J.M. do Ó) DEPARTMENT OF MATHEMATICS, FEDERAL UNIVERSITY OF PARAÍBA
 58051-900, JOÃO PESSOA-PB, BRAZIL
E-mail address: jmbo@pq.cnpq.br

(D. Ferraz) DEPARTMENT OF MATHEMATICS, FEDERAL UNIVERSITY OF PARAÍBA
 58051-900, JOÃO PESSOA-PB, BRAZIL
E-mail address: diego.ferraz.br@gmail.com